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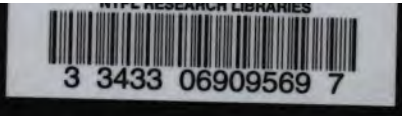
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PREFACE.

THE applications of Mathematical Science, to the Mechanic Arts, have received special attention within the past few years; and no system of instruction, in any department, is now complete, unless it extends beyond the theoretical and into the field of Practical Knowledge.

Geometry, in its essence and structure, is more purely abstract than any other branch of science. In its logical structure, it is conversant only about space; and yet, the relations and principles which it develops, afford the only basis of the true Practical.

To extend Geometry to some of its most interesting and useful applications, an Appendix has been prepared and added to Legendre, embracing many Problems of Geometrical construction, and many applications of Algebra to Geometry: the whole being designed to explain and illustrate the methods of making Geometry a practical as well as a theoretical science.

It would be unjust to those giving instruction, to add to their daily labors, the additional one, of finding appropriate solutions to so

many difficult problems: hence, a Key has been made for their special use, in which the best methods of construction and solution are fully given.

It is confidently hoped that this addition to a work which has been an accepted Text Book, both in this country and in Europe, for many years, may add something to its great value; and it would be pleasant to indulge the hope that it will be received with a portion of that great favor which has been extended to the original work.

FISHKILL-ON-HUDSON,
March, 1875.

NOTE.—In revising the Legendre it was thought proper, while retaining the large body of excellent examples constituting the Appendix referred to in the foregoing Preface, to insert, throughout the work, Exercises intended to accustom students, early in and throughout their course, to make for themselves practical application of geometric principles. Accordingly, at the end of each Book, except the second, may be found such Exercises, many of them numerical, on the Book or Books preceding—which will, it is hoped, prove an additional and timely aid in instruction.

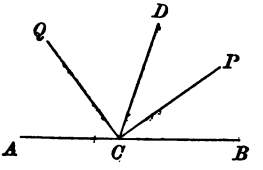
To save the time and labor of teachers already overtaxed, a Key to the Exercises has been prepared for their use, and is made a Supplement of this Key to the Legendre.

COLUMBIA COLLEGE, N. Y.,
SEPT., 1882.

KEY.

PROPOSITION I.—*Show that the bisectrices of two adjacent angles are perpendicular to each other.*

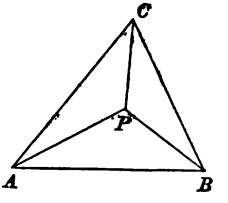
DEMONSTRATION.—Let DCB and DCA be two adjacent angles, and CP and CQ be their bisectrices.



The angle PCD is equal to $\frac{1}{2}$ DCB by definition, and DCQ is equal to $\frac{1}{2}$ DCA: hence, PCD + DCQ = $\frac{1}{2}$ (DCB + DCA). The first member of this equation is equal to the angle QCP, and the second member is equal to a right angle (Bk. I. Prop. 1); hence, the angle QCP is a right angle, *which was to be proved.*

PROP. II.—*Show that the perimeter of any triangle is greater than the sum of the distances from any point within the triangle to its three vertices, and less than twice that sum.*

DEMONSTRATION.—Let ABC be any triangle, P any point within it, and PA, PB, PC, lines from P to the vertices.



In accordance with Bk. I. Prop. 8, we have the following inequalities:

$$AB + BC > PA + PC$$
$$BC + CA > PA + PB$$
$$AB + AC > PB + PC.$$

Adding the first and second members of these inequalities, separately, and dividing by 2, we have,

$$AB + BC + CA > PA + PB + PC,$$

which proves the first part of the proposition.

In accordance with Bk. I, Prop. 7, we have the following inequalities:

$$AB < PA + PB$$
$$BC < PB + PC$$
$$CA < PC + PA.$$

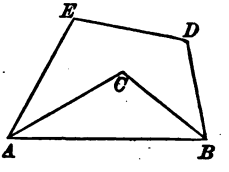
Adding the members of these inequalities, separately, and then factoring, we have,

$$AB + BC + CA < 2\ (PA + PB + PC),$$

which proves the second part of the proposition.

PROP. III.—*Show that the angle between the bisectrices of two consecutive angles of any quadrilateral is equal to one-half the sum of the other two angles.*

DEMONSTRATION.—Let ABDE be any quadrilateral, and let AC and BC be the bisectrices of the angles A and B.



The sum of the 4 angles of the quadrilateral is equal to 4 right angles (Bk. I. Prop. 26, Cor. 1); or, denoting the angles by the letters A, B, D, E, and 1 right angle by R, we have,

$$A + B + D + E = 4R.$$

Dividing both members of this equation by 2, factoring and transposing, we have,

$$\tfrac{1}{2}\ (D + E) = 2R - \tfrac{1}{2}\ (A + B) \quad . \quad . \quad . \quad (1).$$

But CAB and CBA are, respectively, equal to $\frac{1}{2}$ A and $\frac{1}{2}$ B; hence, (Bk. I. Prop. 25, Cor. 1) we have for the angle ACB, denoted by C,

$$C = 2R - \tfrac{1}{2}(A + B) \quad . \quad . \quad . \quad (2).$$

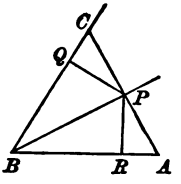
Hence, in accordance with axiom 1, we have, from (1) and (2),

$$C = \tfrac{1}{2}(E + D),$$

which was to be proved.

PROP. IV. THEOREM.—*Show that any point in the bisectrix of an angle is equally distant from the sides of the angle.*

DEMONSTRATION.—Let ABC be any angle, BP its bisectrix, and P any point of BP.



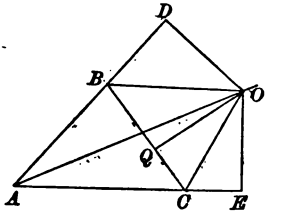
Draw PR and PQ respectively perpendicular to the sides BA and BC; then will PR and PQ be the distances from P, to BA, and BC. In the triangles BPR and BPQ, the angles PBR and PBQ are equal, by hypothesis, and the angles PRB and PQB are equal, because they are both right angles; hence, the angles BPR and BPQ are equal (Bk. I. Prop. 25, Cor. 2). The triangles BPR and BPQ, have the angles PBR and BPR respectively equal to the angles PBQ and BPQ, and the side BP common; hence, they are equal in all their parts (Bk. I. Prop. 6), that is, PR = PQ, *which was to be proved.*

Conversely; If PQ=PR, the line PB is the bisectrix of the angle ABC.

COROLLARY.—If a line, as APC is drawn perpendicular to the bisectrix of an angle, the parts intercepted between it and the sides of the angle will be equal: for the triangles PQC and PRA are equal in all their parts, hence, PC equals PA.

PROP. V.—*If two sides of a triangle are prolonged beyond the third side, show that the bisectrices of the included angle and of the exterior angles, all meet in the same point.*

DEMONSTRATION.—Let ABC be any triangle, and let CO and BO be the bisectrices of the exterior angles ECB and DBC.

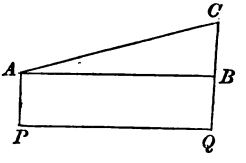


Draw AO, and also, draw OE perpendicular to AE, OD perpendicular to AD, and OQ perpendicular to BC. It is to be shown that AO is the bisectrix of the angle EAD.

Because CO is the bisectrix of ECB, we have, OE = OQ, and because BO is the bisectrix of DBC, OD = OQ; consequently, (Ax. 1) OD = OE. Hence, from the last proposition, AO is the bisectrix of EAD, *which was to be proved.*

PROP. VI.—*Show that the projection of a line on a parallel line is equal to the line itself; and that the projection of a line on a line to which it is oblique, is less than the line itself.*

DEMONSTRATION.—Let PQ be the projection of the parallel line AB, and also the projection of the oblique line AC.

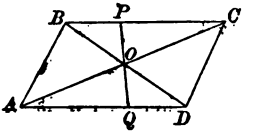


Because AB is parallel to PQ, and because AP and BQ are both perpendicular to PQ and consequently parallel, the figure ABPQ is a parallelogram; hence, AB = PQ, (Bk. I. Prop. 30), *which proves the first part of the proposition.*

Because AB is perpendicular to QC, and AC is oblique to it, AC is greater than AB, (Bk. I. Prop. 15); hence AC is greater than PQ, *which proves the second part of the proposition.*

PROP. VII.—*If a line is drawn through the point of intersection of the diagonals of a parallelogram, show that the line is bisected at the point.*

DEMONSTRATION.—Let ABCD be a parallelogram, AC and BD its diagonals, and O their point of intersection.

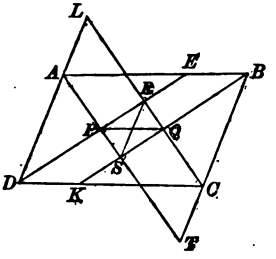


Draw the line PQ, through O, and limited by BC and AD.

In the triangles POC and QOA, the sides AO and OC are equal (Bk. I. Prop. 31), the angles OCP and OAQ are equal (Bk. I. Prop. 20, Cor. 2), and the angles POC and QOA are equal, because they are opposite or vertical; hence, the triangles are equal in all their parts (Bk. I. Prop. 6): consequently, OP = OQ, *which was to be proved.*

PROP. VIII.—*The bisectrices of the four angles of any parallelogram form, by their intersection, a rectangle whose diagonals are parallel to the sides of the given parallelogram.*

DEMONSTRATION.—Let ABCD be any parallelogram, and let AS, BS, CR, and DR be the bisectrices of its angles.



Draw the diagonals PQ, RS, and prolong DA and BC to meet the bisectrices CR and AS, in L and T.

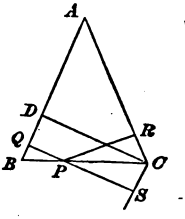
Because AS and BS are the bisectrices of the angles A and B, their included angle is equal to one-half the sum of the angles D and C (Prop. 3, Key); but the sum of D and C is two right angles; hence, the angle PSQ is a right angle. In like manner it may be shown

that each of the angles SQR, QRP and RPS is a right angle; hence, PRQS is a rectangle.

Because DE and BK are parallels included between parallels, they are equal to each other (Bk. I. Prop. 28, Cor. 1). Since AP bisects the angle DAE and is perpendicular to DE, it divides the latter line into two equal parts, that is, EP is one-half of ED; in like manner it may be shown that BQ is one-half of BK; consequently EP is equal to BQ; hence, PQ is parallel to EB (Bk. I. Prop. 30). In the same way it may be shown that RS is parallel to LD, *which was to be proved.*

PROP. IX. THEOREM.—*Show that the sum of the distances from any point in the base of an isosceles triangle to the other two sides, is equal to the distance from the vertex of either angle at the base, to the opposite side.*

DEMONSTRATION.—Let ABC be an isosceles triangle and let P be any point in its base.



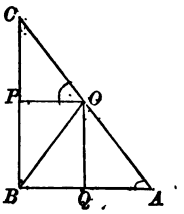
Draw CS parallel to AB, CD and QP perpendicular to AB, PR perpendicular to AC, and prolong QP to S.

The angle PCS is equal to ABC, because they are alternate angles, and the angle PCR is equal to ABC by hypothesis; hence, the angles PCS and PCR are equal. In the triangles PCS and PCR, the angles PCS and PCR are equal, as just shown, the angles PRC and PSC are equal, because both are right angles, and consequently the remaining angles CPR and CPS are equal: these triangles also have the side CP common; hence, they are equal in all their parts (Bk. I. Prop. 6); PR is therefore equal to PS, and QP + PR is equal to QP + PS, or

to QS; but QS is equal to DC (Bk. I. Prop. 28, Cor. 2); hence, PQ + PR = CD, *which was to be proved.*

PROP. X.—*Show that the middle point of the hypothenuse of a right-angled triangle is equally distant from the three vertices.*

DEMONSTRATION.—Let ABC be a right-angled triangle, and let O be the middle point of its hypothenuse.

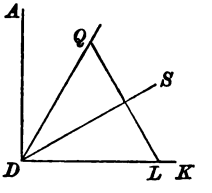


Draw OB; also draw OP parallel to AB, and OQ parallel to CB.

In the triangles OQA and CPO, the angles COP and OAQ are equal (Bk. I. Prop. 19, Sch.); the angles OCP and AOQ are equal for the same reason, and the sides CO and OA are equal by hypothesis; hence, OP is equal to QA; but OP is also equal to BQ, and consequently BQ is equal to QA.—In the triangles OQA and OBQ, the side OQ is common, BQ is equal to QA, and the included angles OQB and OQA are equal because both are right angles; hence, OB is equal to OA, and to its equal OC, *which was to be proved.*

PROP. XI.—*Draw two lines that shall divide a given right angle into three equal parts.*

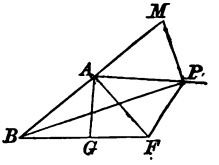
SOLUTION.—Let ADK be the given right angle. Take any distance DL, and on it as a side, construct an equilateral triangle DLQ, and bisect the angle QDL by the line DS, (Bk. III. Probs. 10 and 5). The angle QDL is equal to two-thirds of a right angle (Bk. I. Prop. 25, Cor. 5); hence, KDS and its equal SDQ, are each equal to one-



third of a right angle and the remaining angle QDA is also equal to one third of a right angle. The required lines are, therefore, DS and DQ.

PROP. XII.—*Draw a line AP through the vertex A of a triangle ABF and perpendicular to the bisectrix of the angle A; construct a triangle PBF having its vertex P on AP and its base coinciding with that of the given triangle: then show that the perimeter of PBF is greater than that of ABF.*

DEMONSTRATION.—Let ABF be the given triangle, AG the bisectrix of the angle A, AP perpendicular to AG, and let P be any point of AP. Prolong BA and make AM equal to AF; draw MP.



Because the angles BAF and FAM are adjacent, and AG is the bisectrix of the former, AP must be the bisectrix of the latter (Prop. 1, Key); hence the angles FAP and PAM are equal. In the triangles FAP and PAM, the side AP is common, AM is equal to AF by construction, and the included angles FAP and PAM are equal; hence, PM is equal to PF.—In accordance with Bk. I. Prop. 7, we have the inequality:

$$BP + PM > BA + AM.$$

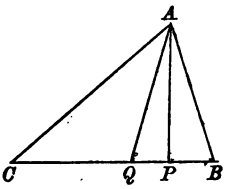
Substituting PF for its equal PM, AF for its equal AM, and adding FB to each member, we have,

$$BP + PF + FB > BA + AF + FB,$$

which was to be proved.

PROP. XIII.—*Let an altitude of the triangle ABC be drawn from the vertex A, and also the bisectrix of the angle A; then show that their included angle is equal to half the difference of the angles B and C.*

DEMONSTRATION.—Let ABC be a triangle, AQ the bisectrix of the angle A, and AP the altitude drawn from the vertex A. The angle CAB is equal to two right angles diminished by the sum of the angles B and C; hence, if we denote a right angle by R, we have,



$$QAB = R - \tfrac{1}{2}B - \tfrac{1}{2}C \quad . \quad . \quad . \quad (1).$$

In the right-angled triangle PAB, we have,

$$PAB = R - B \quad . \quad . \quad . \quad (2).$$

Subtracting (2) from (1), member from member, we have,

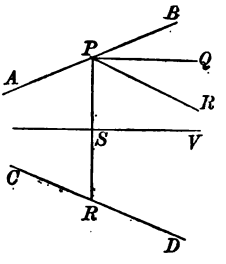
$$QAB - PAB = \tfrac{1}{2}B - \tfrac{1}{2}C, \text{ or } QAP = \tfrac{1}{2}(B - C),$$

which was to be proved.

PROP. XIV.—*Given two lines that would meet if sufficiently prolonged, to draw the bisectrix of their included angle without finding its vertex.*

SOLUTION.—Let AB and CD be the given lines.

Through any point P, of AB, draw PR parallel to CD (Bk. III. Prob. 6). Bisect the angle BPR by the line PQ, and draw the line PR perpendicular to PQ; bisect PR in S (Bk. III. Prob. 1), and draw SV parallel to PQ.

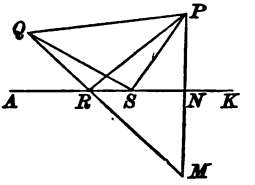


Because CD is parallel to PR, the angle between the given lines is equal to the angle RPB, and consequently the required bisectrix is parallel to PQ; hence, PR is perpendicular to that bisectrix. Since S is the middle

point of PR, and SV is perpendicular to PR, it must be the required bisectrix (Prop. IV, Key, Cor.).

PROP. XV.—*From two points on the same side of a given line, draw two lines that shall meet each other at some point of the given line and make equal angles with that line.*

SOLUTION.—Let P and Q be the given points and AK the given line.

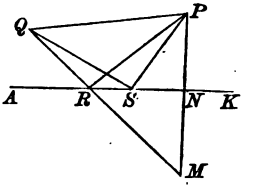


Draw PN perpendicular to AK, and prolong it to M, making NM = NP; draw MQ, and from the point R in which it intersects AK, draw RP.

The right-angled triangles RNP and RNM, have the side PN equal to NM, by construction; the side RN common and the included angles PNR and MNR are equal; hence, the angle NRP is equal to the angle NRM; but the angle ARQ is equal to its opposite angle NRM; hence, the lines PR and QR make equal angles with AK; consequently they are the required lines.

PROP. XVI.—*Show that the sum of the lines drawn from two given points to any point of a given line, is the least possible when these lines are equally inclined to the given line.*

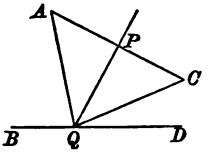
DEMONSTRATION.—Employing the same construction as in the last figure, let P and Q be the given points, and PR and QR equally inclined to AK; also PS and SQ unequally inclined.



The line AK is drawn through the vertex R of the triangle QRP and perpendicular to the bisectrix of the angle QRP; hence, from Prop. XII, Key, QR + RP < QS + SP, *which was to be proved.*

PROP. XVII.—*From two given points, on the same side of a given line, draw two lines meeting on the given line, and equal to each other.*

SOLUTION.—Let A and C be the given points and BD the given line.

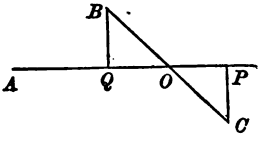


Draw AC, and bisect it by a perpendicular PQ; from the point Q, in which this meets the given line, draw QA and QC.

Because PQ is perpendicular to AC, at its middle point, Q is equidistant from A and C (Bk. I. Prop. 16); hence, AC and CQ are the required lines.

PROP. XVIII.—*From a given point* A, *draw a line that shall be equally distant from two given points* B *and* C.

SOLUTION.—Let A be the first point, and let B and C be the other given points.

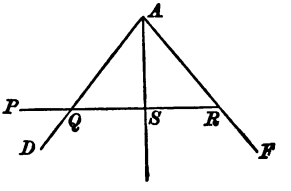


Draw BC, bisect it in O, and then draw AO. From B and C, draw BQ and CP perpendicular to AO.

The triangles BOQ and COP are equal in all their parts; hence BQ equals CP. The line AO is therefore the required line.

PROP. XIX.—*Through a given point draw a line cutting the sides of a given angle and making the interior angles equal to each other.*

SOLUTION.—Let P be the given point and DAF the given angle.



Draw the bisectrix AS, of the given angle, and through P draw PR perpendicular to AS, cutting the sides of the angle in Q and R.

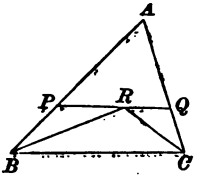
In the triangles ASQ and ASR, we have, AS common, the angles SAQ and

SAR, equal by construction, and the angles ASQ and ASR equal, because they are right angles; hence, these triangles are equal in all their parts. The angles SQA and SRA, are therefore equal, and consequently PR is the required line.

PROP. XX.—*Draw a line* PQ *parallel to the base* BC *of a triangle* ABC, *so that* PQ *shall be equal to the sum of* BP *and* CQ.

SOLUTION.—Let ABC be the given triangle.

Draw the bisectrices BR and CR, of the angles B and C, and through their intersection R, draw PQ parallel to BC.



Since PQ is parallel to BC, the angles QRC and RCB are equal; but, RCQ and RCB are equal by construction; hence, the angles QRC and QCR are equal; the triangle RQC is therefore isosceles (Bk. II. Prop. 12), and consequently RQ is equal to QC. In like manner it may be shown that RP is equal to PB. Hence,

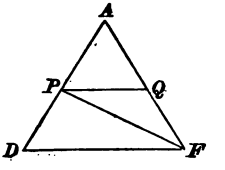
$$PR + RQ \text{ or } PQ = BP + QC;$$

the line PQ is therefore the required line.

PROP. XXI.—*In a given isosceles triangle draw a line that shall cut off a trapezoid whose base is the base of the given triangle and whose other sides shall be equal to each other.*

SOLUTION.—Let ADF be the given isosceles triangle.

Draw the bisectrix FP of the angle F, and through the point in which it meets AD draw PQ parallel to DF.

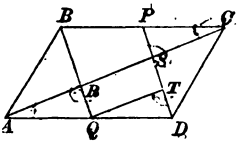


It may be shown as in the last problem, that the triangle PQF is isosceles, the side

PQ being equal to QF. Since PQ is parallel to DF, the angles AQP and APQ are equal, and consequently AP is equal to AQ; AD — AP or PD, is equal to AF — AQ or QF. Hence, PQ = QF = PD, and consequently PQ is the required line.

PROP. XXII.—*If two opposite sides of a parallelogram are bisected and lines be drawn from the points of bisection to the vertices of the opposite angles, show that these lines will divide the diagonal which they intersect, into three equal parts.*

DEMONSTRATION.—Let ABCD be any parallelogram, and let P and Q be the middle points of BC and AD.

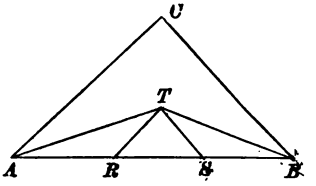


Draw PD and BQ, also draw QT parallel to AC.

Since BP is parallel and equal to QD, BQ and PD are also parallel and equal (Bk. I. Prop. 30). In the triangles ARQ and CSP, the angles ARQ and CSP are equal because they are alternate exterior angles with respect to BQ and PD. The angles RAQ and SCP are equal, because they are alternate angles, and the sides AQ and CP are equal because they are halves of equal lines (Ax. 7); hence, the triangles are equal in all their parts: AR is therefore equal to SC. In the triangles ARQ and QTD, the angles ARQ and QTD are equal because their sides are parallel and lie in the same direction (Bk. I. Prop. 24); the angles RAQ and TQD are equal because they are opposite exterior and interior angles, with respect to BQ and PD, and the sides AQ and QD are equal by hypothesis; hence the triangles are equal in all their parts: QT is therefore equal to AR: but QT is equal to RS; hence, QT = AR = SC, *which was to be proved.*

PROP. XXIII.—*Construct a triangle, having given the two angles at the base and the sum of the three sides.*

SOLUTION.—Let AB be equal to the sum of the three sides, and let the angles BAC and ABC be equal to the angles at the base of the required triangle.

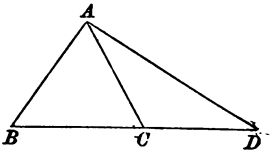


Draw the bisectrices AT and BT, of the angles A and B, and through their point of intersection T, draw TR parallel to CA or TS parallel to CB.

The angles RTA and TAC are equal because they are alternate; but TAC is equal to RAT by construction; hence, the triangle ART is isosceles, and consequently AR = RT. In like manner BS = ST. The angle TRS is equal to CAS, the angle TSR is equal to CBA and the sum of RT, RS, and ST is equal to AB; hence, TRS is the required triangle.

PROP. XXIV.—*Construct a triangle, having given one angle, one of its including sides and the sum of the other two sides.*

SOLUTION.—Let ABC be the given angle, AB one of its including sides, and let BD be equal to the sum of the other two sides.

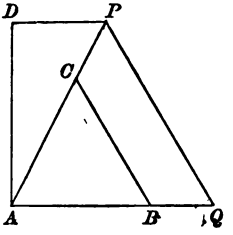


Draw AD. At A make the angle DAC equal to the angle ADC (Bk. III. Prob. 4).

The triangle ACD is isosceles and consequently, AC = CD (Bk. I. Prop. 12). Hence, BC + CA = BC + CD = BD. The triangle ABC is therefore the required triangle.

PROP. XXV.—*Construct an equilateral triangle, having given one of its altitudes.*

SOLUTION.—Let AD be equal to the given altitude. Draw any line AQ, perpendicular to AD, and on any part of it, as AB, construct an equilateral triangle ABC. Through D, draw DP parallel to AQ, intersecting AC, produced in P. Through P draw PQ parallel to CB: then will APQ be the required triangle.

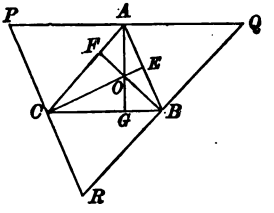


For, the triangles ACB and APQ, being similar, and ACB equilateral, APQ is also equilateral. But the altitude of the triangle APQ is equal to the given altitude: hence, APQ is the required triangle.

PROP. XXVI.—*Show that the three altitudes of a triangle all intersect in a common point.*

DEMONSTRATION.—Let ABC be the given triangle.

Through A, draw PQ parallel to CB, through B, draw RQ parallel to CA, and through C, draw PR parallel to AB. Through A, B and C, draw perpendiculars to PQ, QR, and RP, meeting the sides of the given triangle in G, F, and E.



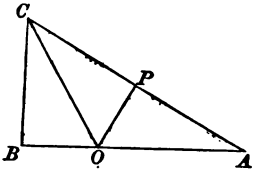
Now, BQ and CA are equal, because they are parallels between parallels; for a like reason RB and AC are equal: hence, B is the middle point of QR. In like manner, C is the middle point of RP, and A is the middle point of PQ. Because PQ, QR, and RP are chords of the circle that circumscribes the triangle PQR, the perpendiculars, AG, BF, and CE, pass through the centre O

of that circle: but these perpendiculars are also the altitudes of the given triangle; hence, these altitudes all intersect at the same point, *which was to be proved.*

PROP. XXVII.—*If one of the acute angles of a right-angled triangle is double the other, show that the hypothenuse is double the smaller side about the right angle.*

DEMONSTRATION.—Let ABC be a right-angled triangle in which the angle C is double the angle A.

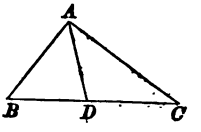
Draw the bisectrix CO of the angle C, and from the point O in which it meets BA, draw OP perpendicular to the hypothenuse CA.



Because the angle C is equal to twice the angle A, we have the angle OCA equal to the angle OAC; that is, the triangle OCA is isosceles; hence, the perpendicular OP, bisects the side CA; consequently CP is equal to one-half of the hypothenuse. But CP is equal to CB, because the right-angled triangles CPO and CBO have the same hypothenuse CO, and the acute angle OCP in the one, equal to the acute angle OCB in the other. Hence, CA = 2CP = 2CB, *which was to be proved.*

PROP. XXVIII.—*Let a median be drawn from the vertex of any angle* A *of a triangle* ABC; *then show that the angle* A *is a right angle when the median is equal to half the side* BC, *an acute angle when the median is greater than half of* BC, *and an obtuse angle when it is less than half of* BC.

DEMONSTRATION.—Let ABC be a triangle, and let AD be the median drawn from the vertex A.



1°. Suppose AD = BD = DC.

The triangle ADC is isosceles, because its sides AD and DC are equal; hence, the angles DAC and DCA are equal. In like manner it may be shown that the angles DAB and DBA are equal. Hence, the sum of the angles CAD and DAB, or the angle BAC, is equal to the sum of the angles ABD and ACD, that is, the angle at A is equal to half the sum of the three angles of the given triangle, or to one right angle, *which was to be proved.*

2°. Suppose AD > BD and consequently AD > CD.

Since, AD > BD, we have from the converse of Prop. 9, Bk. I, the angle B greater than the angle BAD. In like manner the angle C can be shown to be greater than the angle DAC. Hence, the sum of the angles BAD and DAC, or the angle at A, is less than half the sum of the three angles of the given triangles, and consequently it is acute, *which was to be proved.*

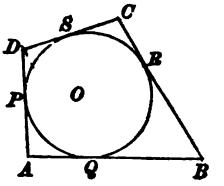
3°. Suppose AD < BD, and consequently AD < DC.

It follows from the converse of Prop. 9, Bk. I, that the angle B is less than the angle BAD; and the angle C is less than the angle CAD; hence, the angle at A is greater than half the sum of the three angles of the given triangle, that is, it is obtuse, *which was to be proved.*

PROP. XXIX.—*Let any quadrilateral be circumscribed about a circle: then show that the sum of two opposite sides is equal to the sum of the other two opposite sides.*

DEMONSTRATION.—Let ABCD be any quadrilateral circumscribed about the circle O, and let its sides be tangent to this circle at the points P, Q, R, and S.

From the principle demonstrated in the Corollary to Prob. 14, Bk. III, we have the following equations:



$$DS = DP$$
$$CS = CR$$
$$AQ = AP$$
$$BQ = BR.$$

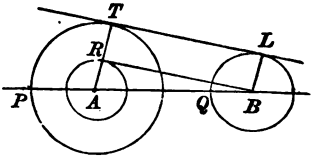
Adding these equations, member to member, and recollecting that $DS + CS = DC$, $AQ + BQ = AB$, $DP + AP = DA$, and $CR + BR = CB$, we have,

$$DC + AB = DA + CB,$$

which was to be proved.

PROP. XXX.—*Draw a straight line tangent to two given circles.*

SOLUTION.—Let A and B be the centres of the two circles, and let AP and BQ be their radii, and let $AP > BQ$.



Draw the line AB, through their centres.

With a radius AR, equal to $AP - BQ$, and a centre A, describe the circle AR; through B draw BR tangent to this auxiliary circle, and let AR be drawn through the point of contact R, and prolonged to T; draw the radius BL, parallel to AT, and through the points T and L draw the line TL.

Because AR is perpendicular to BR (Bk. III, Prop. 9), and BL is parallel to AR, the angles TRB and LBR are right angles. But, RT is equal to BL, by construction; hence, the figure TRBL is a rectangle, and consequently the line TL is perpendicular to the radii

AT and BL, at their extremities; TL is therefore tangent to both the given circles; hence, TL is the required line.

SCHOLIUM.—A second tangent may, in like manner, be constructed on the other side of the circles, and it may be shown that both intersect the line of centres at a common point, to the right of B.

By using an auxiliary circle whose radius is equal to AP + BQ, two other tangents may be constructed intersecting each other on the line of centres, between A and B. The first pair of tangents is said to be external, the second pair is said to be internal.

There may be several positions of the given circles.

1°. If the circles are external to each other, they will always have two external and two internal tangents, common to both.

2°. If the circles touch each other externally, they will have two external tangents and only one internal tangent.

3°. If the circles cut each other, they will have two external tangents but no internal tangent.

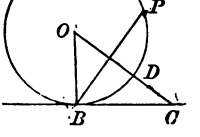
4°. If the circles touch each other internally, they will have one external but no internal tangent.

5°. If one circle lies wholly within the other, they can have no common tangent whatever.

PROP. XXXI.—*Through a given point* P, *draw a circle that shall be tangent to a given line* CB, *at a given point* B.

SOLUTION.—Let P be a given point, CB a given line, and B a given point on that line.

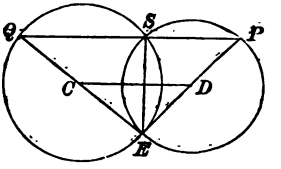
Draw PB and bisect it by a perpendicular OC, meeting BC in C; at B erect BO perpendicular to CB, and meeting OC in O. Then, with O as a centre, and OB as a radius describe a circle.



Because OD is perpendicular to PB, at its middle point, O is equally distant from B and P (Bk. I, Prop. 16); hence, the circle passes through P. Because BC is perpendicular to the radius OB, at its extremity, it is tangent to the circle at B. Hence, the circle BDP is the required circle.

PROP. XXXII.—*Let two circles intersect each other, and through either point of intersection let diameters of the circles be drawn: then show that the other extremities of these diameters and the other point of intersection, lie in the same straight line.*

DEMONSTRATION.—Let C and D be the centres of two circles which intersect each other at E and S, and let EQ and EP be the diameters of the circles through E.



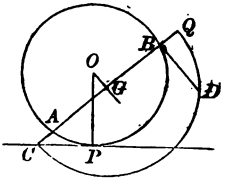
Draw QS, PS, and SE.

Because ECQ is a diameter of the circle C, the angle QSE is a right angle (Bk. III, Prop. 18, Cor. 2); in like manner the angle ESP is a right angle: hence, the line ES meets QS and SP, making the sum of the angles ESP and ESQ equal to two right angles; the lines QS and SP therefore form a single straight line QSP (Bk. I, Prop. 8), *which was to be proved.*

PROP. XXXIII.—*Through two given points,* A *and* B, *draw a circle that shall be tangent to a given line* CP.

SOLUTION.—Let A and B be the given points and CP the given line.

Draw AB, and prolong it till it intersects CP at C; lay off BQ equal to AC, and on CQ, as a diameter, construct a semicircle QDC; draw BD perpendicular to CQ, cutting this semi-

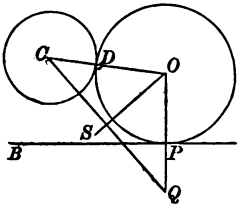


circle in D; make CP = BD and draw PO perpendicular to CP; also draw GO perpendicular to, and bisecting AB; from the point O, in which the last two lines intersect, as a centre, and with a radius equal to OP draw a circle BPA.

The circle BPA is tangent to the line PC, at P, because CP is perpendicular to the radius OP, at its extremity. Since BD is perpendicular to CQ, it is a mean proportional between CB and BQ, that is, between CB and CA (Bk. IV, Prop. 23); hence, the points A and B are on the circle APB (converse of Prop. 30, Bk. IV): consequently the circle BPA is the required circle.

PROP. XXXIV.—*Draw a circle that shall be tangent to a given circle* C *and also to a given line* BP, *at a given point* P.

SOLUTION.—Let C be the given circle, BP the given line, and P the given point.

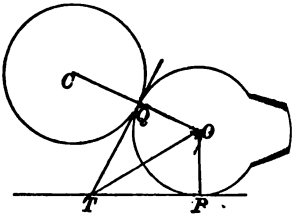


Through P draw the line QO, perpendicular to BP, and make PQ equal to the radius CD of the circle C; draw QC, and bisect it by a perpendicular SO; from the point O, in which SO and QO intersect, as a centre, and with a radius OP, describe the circle PD.

Because SO is perpendicular to QC, at its middle point, OQ = OC and consequently OP = OD. But BP is tangent to the circle O, at P, because it is perpendicular to the radius OP at its extremity. The circle O is tangent to the circle C, because the distance OC = OD + DC (Bk. III, Prop. 14, Scho.). Hence, the circle O is the required circle.

PROP. XXXV.—*Draw a circle that shall be tangent to a given line*, TP, *and also to a given circle* C, *at a given point* Q.

SOLUTION.—Let TP be the given line, C the centre of the given circle, and let Q be any given point on the circle C.

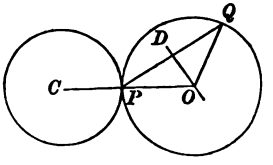


Through Q draw the line CQO, and through the same point Q draw QT perpendicular to CO; draw the bisectrix of the angle QTP, meeting CO in O; draw OP perpendicular to TP, and then with O as a centre and with OP as a radius, describe the circle ⬭.

Since O is on the bisectrix TO, OP and OQ are equal; hence, the circle whose radius is OP, passes through Q, and is tangent to the circle C at Q; it is also tangent to TP; hence, it is the required circle.

PROP. XXXVI.—*Draw a circle that shall pass through a given point* Q, *and be tangent to a given circle* C, *at a given point* P.

SOLUTION.—Let Q be the given point, C the centre of the given circle, and P the given point on that circle.

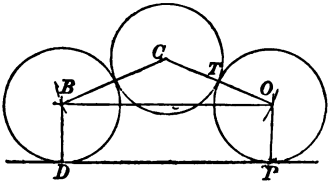


Draw PQ, and bisect it by the perpendicular DO; draw CP, and prolong it till it meets DO, in O; with O as a centre and OP as a radius, describe the circle OP.

Because DO is perpendicular to PQ, at its middle point, OQ = OP; hence, the circle OP passes through Q. Because CO = CP + PO the circle OP is tangent to the circle C. Hence, the circle OP is the required circle.

PROP. XXXVII.—*Draw a circle, with a given radius, that shall be tangent to a given line* DP, *and to a given circle* C.

SOLUTION.—Let DB be the given radius, DP the given line, and C the centre of the given circle, and CT its radius.

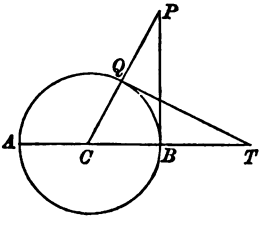


Draw a line BO, parallel to DP, and at a distance from it equal to DB. With C as a centre, and a radius CO, equal to CT + DB, describe an arc cutting BO in the points B and O. With B as a centre, and with a radius equal to BD, describe a circle; also with O as a centre, and the same radius, describe a second circle.

These circles are tangent to the line DP, at the points D and P; they are also tangent to the given circle, because the distance from C, to the centre of either, is equal to the sum of the radii CT and BD; hence, either of the circles B and O, is the required circle.

PROP. XXXVIII.—*Find a point, in the prolongation of any diameter of a given circle, such that a tangent from it to the circumference shall be equal to the diameter of the circle.*

SOLUTION.—Let C be the centre of the given circle, and let AB be any diameter.

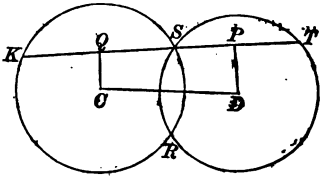


At B, draw BP perpendicular to AB and equal to the diameter AB; draw PC, and at the point Q, in which it cuts the circumference, draw the tangent QT and prolong it till it meets the prolongation of AB, in T.

The triangles CQT and CBP have the common angle PCT; the angles CQT and CBP are equal, because they are right angles; and the sides CQ and CB are equal, because they are radii of the same circle: hence, these triangles are equal in all their parts, and consequently, QT = BP = AB; T is therefore the required point.

PROP. XXXIX.—*Show that when two circles intersect each other, the longest common secant that can be drawn through either point of intersection is parallel to the line joining the centres of the circles.*

DEMONSTRATION.—Let C and D be the centres of two circles intersecting each other at S and R, and let KT be a common secant drawn through one of the points of intersection S.



Draw CD; and from C and D let fall CQ and DP perpendicular to KT, and meeting it in the points Q and P.

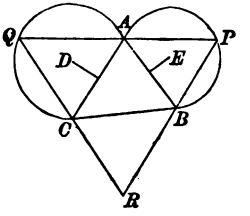
The distances KQ and QS are equal (Bk. III, Prop. 6), as are also the distances SP and PT; hence, KT = 2QP: but QP is the projection of CD on the line KT, and this projection is the greatest possible when KT is parallel to CD (Prop. VI, Key). Hence, the whole line KT is the greatest possible when it is parallel to CD, *which was to be proved.*

PROP. XL. PROBLEM.—*Construct the greatest possible equilateral triangle whose sides shall pass through three given points,* A, B, *and* C, *not in the same straight line.*

SOLUTION.—Let A, B, and C be the three given points.

Join the three points, forming the triangle ABC; on the sides AB and AC, as chords, construct segments of circles APB and AQC capa-

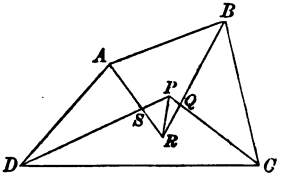
ble of containing an angle equal to two-thirds of a right angle (Bk. III, Prob. 16); through A draw QAP parallel to the line through D and E, the centres of the segments; from the points Q and P, draw QC and PB, and prolong them till they meet at R.



Any triangle whose sides pass through A, B, and C, and whose two vertices are in the arcs APB and AQC, is equilateral, because each of its angles is equal to two-thirds of a right angle; but no other triangle whose sides pass through A, B, and C, can be equilateral. Now, of the group of equilateral triangles that can be formed, as just indicated, that one will be the greatest possible which has one of its sides the greatest possible; but from Prop. XXXIX, Key, the side through A will be greatest possible when it is parallel to the line joining D and E: hence, QPR is the required triangle.

PROP. XLI.—*Show that the bisectrices of the four angles of any quadrilateral intersect in four points, all of which lie on the circumference of the same circle.*

DEMONSTRATION—Let ABCD be any quadrilateral, and let AR, BR, CP and DP, be the bisectrices of its four angles. Draw PR.

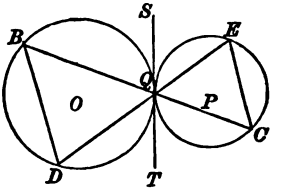


The angle PSR, or its equal ASD, is equal to one-half the sum of ABC and DCB (Prop. III, Key). In like manner it may be shown that the angle PQR is equal to one-half the sum of the angles BAD and CDA. Hence, the sum of the angles PSR and PQR is equal to one-half the

sum of the four angles of the given quadrilateral, that is, to two right angles. Consequently, from the converse of Prop. 18, Bk. III, Cor. 4, the quadrilateral SPQR can be inscribed in a circle, *which was to be proved.*

PROP. XLII.—*If two circles touch each other externally, and if two common secants are drawn through the point of contact and terminating in the concave arcs, show that the lines joining the extremities of these secants, in the two circles, are parallel.*

DEMONSTRATION.—Let O and P be the centres of two circles which are tangent to each other at the point Q.



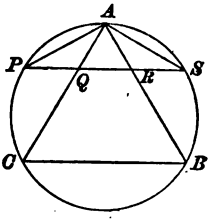
Draw the secants BC and DE, terminating in the concave arcs BD and EC; draw also BD and EC, and at Q draw TS tangent to both circles.

The angle SQE is equal to QCE (Bk. III, Prop. 21 and 18); in like manner the angle TQD is equal to QBD. But SQE and TQD are equal because they are opposite angles; hence, QCE and QBD are equal, and consequently, the lines CE and BD are parallel (Bk. I, Prop. 19, Cor. 1); *which was to be proved.*

PROP. XLIII.—*Let an equilateral triangle be inscribed in a circle, and let two of the subtended arcs be bisected by a chord: then show that the sides of the triangle divide the chord into three equal parts.*

DEMONSTRATION.—Let ABC be an equilateral triangle inscribed in a circle ASBCP, and let P and S be the middle points of the arcs APC and ASB.

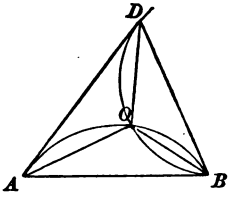
Draw PS, cutting AC and AB in Q and R; also draw AP and AS.



The angles ASR and RAS are equal because they are measured by halves of the equal arcs AP and SB; hence, AR = RS. In like manner it may be shown that AQ = PQ. In the triangle AQR, the angle ARQ is measured by $\frac{1}{2}$ (SB + PA), the angle AQR is measured by $\frac{1}{2}$ (AS + PC), and the angle QAR is measured by $\frac{1}{2}$CB, hence, the three angles are equal, and consequently QR = AQ = AR, or PQ = QR = RS, *which was to be proved.*

PROP. XLIV.—*Find a point within a triangle, such that the angles formed by drawing lines from it to the three vertices of the triangle, shall be equal.*

SOLUTION.—Let ABD be the given triangle. On AB construct an arc that will contain an angle equal to four-thirds of a right angle; on BD construct another arc that will contain an angle equal to four-thirds of a right angle, and intersecting the first at O; draw OD, OA, and OB.

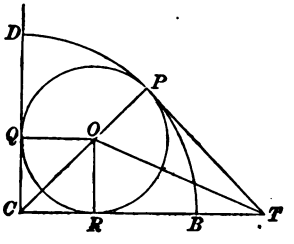


The angles AOD and DOB are each equal to four-thirds of a right angle, and consequently, the angle AOB is also equal to four-thirds of a right angle; hence, O is the required point.

PROP. XLV.—*Inscribe a circle in a quadrant of a given circle.*

SOLUTION.—Let CBPD be the given quadrant.

Draw the bisectrix CP of the angle DCB, and at P draw a tangent PT to the quadrant, meeting CB produced at T.



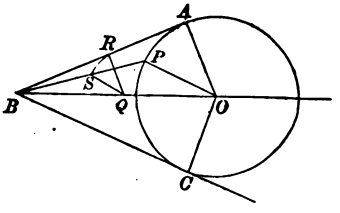
Draw the bisectrix TO, of the angle CTP, and from the point O, in which it meets CP, draw OR and OQ perpendicular to CB and CD; with O as a centre, and OR as a radius, describe a circle.

Because TO is the bisectrix of the angle RTP, OR = OP (Prop. IV, Key); because CP is the bisectrix of the angle BCD, OR = OQ: hence, the circle OR passes through P and Q. This circle is tangent to CT because CT is perpendicular to OR at R, and for a like reason it is also tangent to CD at Q; the circle OR is tangent to the arc DPB, because CO = CP — OP: hence, the circle OR is the required circle.

PROP. XLVI.—*Through a given point* P, *within a given angle* ABC, *draw a circle that shall be tangent to both sides of that angle.*

SOLUTION.—Let P be the given point and ABC the given angle.

Draw the bisectrix BO, of the given angle, and also the line BP; from any point Q, of BO, draw QR perpendicular to AB, and from Q, as a centre, and with a radius QR, draw the arc RS, cutting BP in S; draw SQ and also draw PO parallel to SQ, intersecting BO at O; then draw OA parallel to QR, and with O as a centre and OA as a radius, describe the circle APC: also draw OC perpendicular to BC.



The triangles BSQ and BPO have the angle at B common, and since QS is parallel to OP, the angles BQS and BOP are equal, as are also the angles BSQ and BPO; hence, the triangles are similar (Bk. IV, Prop. 18), and consequently, their corresponding sides are proportional,

We have, therefore, the proportion

$$BQ : BO :: QS : OP \quad . \quad . \quad . \quad (1).$$

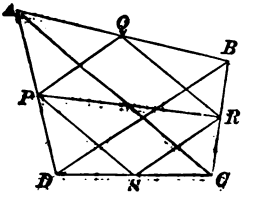
In like manner, we have from the triangles BQR and BOA, the proportion,

$$BQ : BO :: QR : OA \quad . \quad . \quad . \quad (2).$$

Because the first three terms of proportions (1) and (2) are equal, each to each, their fourth terms must be equal, that is, OP = OA. Hence, the circle whose radius is OA, passes through P. The distance OC is equal to OA (Prop. IV, Key); hence, the circle whose centre is O passes through C. Furthermore, BA is perpendicular to OA, at its extremity, and BC is perpendicular to OC, at its extremity; hence, the circle whose radius is OA, is tangent to both sides of the given angle; it is therefore the required circle.

PROP. XLVII.—*Show that the middle points of the sides of any quadrilateral are the vertices of an inscribed parallelogram.*

DEMONSTRATION.—Let ABCD be any quadrilateral, and let P, Q, R, and S, be the middle points of its four sides.



Draw PQ, QR, RS, and SP; also draw the diagonals BD and AC.

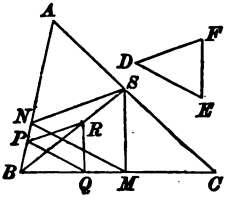
Because AQ = QB, and AP = PD, we have the proportion,

$$AP : PD :: AQ : QB.$$

Hence, from Bk. IV, Prop. 16, PQ is parallel to DB. In like manner it may be shown that RS is parallel to DB: hence, PQ and SR are parallel to each other. The lines QR and PS are parallel to AC, for similar reasons; hence, they are parallel to each other, and consequently, the figure PQRS is a parallelogram inscribed in the given quadrilateral, *which was to be proved.*

PROP. XLVIII.—*Inscribe in a given triangle, a triangle whose sides shall be parallel to the sides of a second given triangle.*

SOLUTION.—Let ABC be the first given triangle, and let DEF be the second given triangle.



Take any point P, on the side AB, and through it draw PQ parallel to DE; from Q draw QR parallel to EF, and from P draw PR parallel to DF, intersecting PR in R; draw BR and prolong it to S. Through S draw SM parallel to RQ, and SN parallel to RP, and then join M and N.

From the triangles BSN and BRP, we have (Bk. IV, Prop. 15),

$$BR : RS :: BP : PN \quad . \quad . \quad . \quad (1).$$

From the triangles BSM and BRQ, we have, like reasons,

$$BR : RS :: BQ : QM \quad . \quad . \quad . \quad (2).$$

Combining proportions (1) and (2), we have,

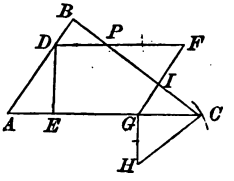
$$BP : PN :: BQ : QM;$$

hence, from (Bk. IV. Prop. 16), we have, NM parallel to PQ. The sides of the inscribed triangle NSM, are therefore parallel to the sides of the triangle PRQ, and consequently, to the sides of the triangle DFE. The triangle NSM is therefore the required triangle.

PROP. XLIX.—*Through a point* P, *within a given angle, draw a line such that it and the parts of the sides that are intercepted shall contain a given area.*

SOLUTION.—Let BAC be the given angle, and P the given point.

Through P, draw DPF parallel to AC, and complete the parallelogram DFGA, in such manner that its area shall be equal to the given area. To do this, first construct a triangle equal to the given area (Bk. IV, Prob. 6), then convert this into an equivalent triangle whose altitude is equal to the altitude DE, and then make DF and AG each equal to one-half the base of the last triangle. From G, draw GH perpendicular to AC and make it equal to PD; with H as a centre, and a radius equal to PF describe an arc cutting AC in C; then draw CP and prolong it to B.



The triangles DBP, PFI, and IGC are similar (Bk. IV, Prop. 21); they are therefore proportional to the squares of their homologous sides DP, PF, and GC; hence, we have the continued proportion

$$\text{PFI} : \text{DBP} : \text{GIC} :: \overline{\text{PF}}^2 : \overline{\text{PD}}^2 : \overline{\text{GC}}^2 \quad . \quad . \quad . \quad (1),$$

whence, by composition, (Bk. II, Prop. 6),

$$\text{PFI} : \text{DBP} + \text{GIC} :: \overline{\text{PF}}^2 : \overline{\text{PD}}^2 + \overline{\text{GC}}^2 \quad . \quad . \quad . \quad (2).$$

But, from the nature of the construction (Bk. IV, Prop. 11), we have,

$$\overline{\text{HC}}^2 \text{ or } \overline{\text{PF}}^2 = \overline{\text{HG}}^2 + \overline{\text{GC}}^2 = \overline{\text{PD}}^2 + \overline{\text{GC}}^2;$$

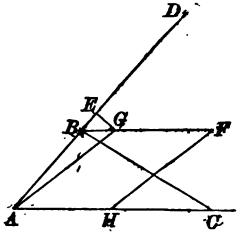
hence, the terms of the second couplet of proportion (2) are equal, and consequently the terms of the first couplet are equal, that is,

$$\text{PFI} = \text{DBP} + \text{GIC}.$$

If now we take away from the parallelogram AF, the triangle PFI and then add to it its equal, DBP + GIC, we shall have the triangle BAC. The triangle BAC is therefore equal to the parallelogram AF, which is equal to the given area; hence, BAC is the required triangle.

PROP. L.—*Construct a parallelogram, whose area and perimeter are respectively equal to the area and perimeter of a given triangle.*

SOLUTION.—Let ABC be the given triangle. Prolong AB, making BD = BC, and bisect AD in E; draw BF parallel to AC; with A as a centre and a radius equal to AE, describe an arc cutting BF in G; draw AG; bisect AC, in H, and complete the parallelogram AHFG.

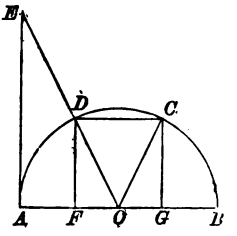


The sides AG and HF of the parallelogram AF, are each equal to AE, and consequently their sum is equal to AD, or AB + BC. The sides AH and GF are each equal to one-half of AC, and consequently their sum is equal to AC; hence, the perimeter of the parallelogram AF is equal to that of the given triangle. The altitude of the parallelogram is equal to that of the given triangle, and the base of the parallelogram is one-half that of the given triangle; hence, the area of the parallelogram is equal to that of the given triangle (Bk. IV, Prop. 5 and 6). The parallelogram AGFH is therefore the required parallelogram.

PROP. LI.—*Inscribe a square in a semicircle, that is, a square, two of whose vertices are in the diameter and the other two in the semicircumference.*

SOLUTION.—Let ADCB be a semicircumference whose diameter is AB, and whose centre is O.

Draw AE perpendicular to, and equal to, AB; draw EO, cutting the semicircumference in D; draw DC parallel to AB; also, draw DF and CG perpendicular to AB; draw OC.

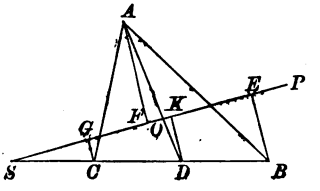


The triangles OAE and OFD are similar; but, OA = ½AE, and consequently OF = ½FD. The triangles OFD and OGC are equal in all their parts; hence, OG = OF, and consequently FG = FD; but FD = GC, and FG = DC; hence, FDCG is the required square.

PROP. LII.—*Through a given point* P, *draw a line cutting a triangle, so that the sum of the perpendiculars to it from the two vertices, on one side of the line, shall be equal to the perpendicular to it, from the vertex, on the other side of the line.*

SOLUTION.—Let P be the given point and ABC the given triangle.

Draw the median AD, and on it lay off DO = ½DA; draw PO, and prolong it till it meets DC prolonged, in S; upon this line let fall the perpendiculars CG, AF, DK and BE.



Because D is the middle of CB, we have the following equations:

$$SD = SC + \tfrac{1}{2}CB$$
$$SD = SB - \tfrac{1}{2}CB.$$

Adding, and dividing by 2, we have,

$$SD = \tfrac{1}{2}(SC + SB) \quad . \quad . \quad . \quad (1).$$

From the similar triangles SCG and SBE, we have,

$$SC : SB :: CG : BE \quad . \quad . \quad . \quad (2);$$

whence, by composition

$$SC : SC + SB :: CG : CG + BE \quad . \quad . \quad . \quad (3).$$

From the similar triangles SCG and SDK, we have

$$SC : SD :: CG : DK \quad . \quad . \quad . \quad (4).$$

Because the antecedents in (3) and (4) are equal, each to each, we have,

$$SC + SB : SD :: CG + BE : DK \quad . \quad . \quad . \quad (5).$$

But, $SC + SB = 2SD$, hence,

$$CG + BE = 2DK \quad . \quad . \quad . \quad (6).$$

From the similar triangles AFO and DKO, we have,

$$AF : DK :: AO : DO \quad . \quad . \quad . \quad (7);$$

but, $AO = 2DO$ by construction, hence,

$$AF = 2DK \quad . \quad . \quad . \quad (8).$$

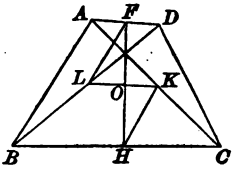
From equations (6) and (8), we have,

$$AF = CG + BE;$$

hence, PS is the required line.

PROP. LIII.—*Show that the line which joins the middle points of two opposite sides of any quadrilateral bisects the line joining the middle points of the two diagonals.*

DEMONSTRATION.—Let ABCD be any quadrilateral, F and H the middle points of two opposite sides; and let L and K be the middle points of the two diagonals BD and AC.



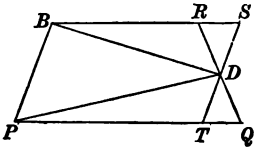
Draw LK, FH, FL and KH.

Because F is the middle point of AD, and L the middle point of BD, FL is

parallel to AB, and equal to $\frac{1}{2}$AB. In like manner it may be shown, that HK is parallel to AB, and equal to $\frac{1}{2}$AB. Hence, HK and LF are parallel and equal to each other. The triangles FOL and HOK, have their angles equal each to each and the sides LF and KH equal; hence, they are equal in all their parts, and consequently, OL = OK, *which was to be proved.*

PROP. LIV.—*If from the extremities of one of the oblique sides of a trapezoid, lines be drawn to the middle of the opposite side, show that the triangle thus formed is equal to one-half the given trapezoid.*

DEMONSTRATION.—Let BPQR be a trapezoid, and let BD and PD be lines drawn from B and P, to D, the middle point of QR.

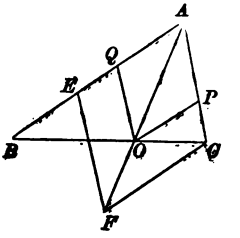


Through D, draw TD parallel to BP, and prolong it till it meets the prolongation of BR, in S.

The triangles DRS and DTQ, have their corresponding angles equal and the sides DR and DQ equal; hence, they are equal in all their parts. If we take the triangle DTQ from the given trapezoid, and then add the equal triangle DRS, we shall have the parallelogram BPTS; hence, this parallelogram is equal to the given trapezoid. The parallelogram PTSB, and the triangle PBD, have a common base BP and the same altitude; viz., the perpendicular distance from D, to PB; hence, from Prop. 5 and 6, Bk. IV, the triangle is equal in area to one-half the parallelogram, and consequently, to one-half the given trapezoid, *which was to be proved.*

PROP. LV.—*Find a point in the base of a triangle, such that the lines drawn from it parallel to, and limited by, the other sides of the triangle, shall be equal to each other.*

SOLUTION.—Let ABC be the given triangle. Lay off AE equal to AC, and complete the parallelogram ACFE. Draw FA cutting BC in O; draw OP parallel to FC, and OQ parallel to FE.



From the similar triangles AFC and AOP, we have,

$$AF : AO :: FC : OP \ldots (1).$$

From the similar triangles AFE and AOQ, we have,

$$AF : AO :: FE : OQ \ldots (2).$$

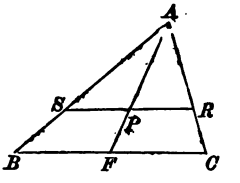
From (1) and (2),

$$FC : OP :: FE : OQ.$$

But, FC = FE; hence OP = OQ; the point O is therefore the required point.

PROP. LVI.—*Show that the line drawn from the middle point of the base, of any triangle, through the middle point of any line of the triangle parallel to the base, will pass through the opposite vertex, if sufficiently produced.*

DEMONSTRATION.—Let ABC be any triangle, SR a line parallel to the base BC, and let F and P, be the middle points of BC and SR.

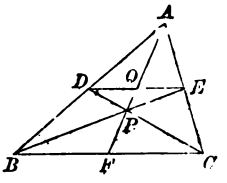


Draw PF, and prolong it.

The straight line drawn from A to F, bisects SR (Bk. IV. Prop. 22), that is, it passes through P; but only one straight line can be drawn through F and P, consequently, the line FP, if sufficiently produced, will pass through A, *which was to be proved.*

PROP. LVII.—*Show that the three medians of any triangle meet in a common point.*

DEMONSTRATION.—Let ABC be any triangle and let BE and CD be two of its medians, intersecting each other in P.



Draw DE; and through F, the middle point of BC, draw FP, cutting DE in Q, and prolong it.

The line DE is parallel to BC; consequently the triangles PFB and PQE are similar, as are also the triangles PFC and PQD. From PFC and PQD, we have,

$$PF : PQ :: FC : QD \quad . \quad . \quad . \quad (1).$$

From the triangles PFB and PQE, we have,

$$PF : PQ :: FB : QE \quad . \quad . \quad . \quad (2).$$

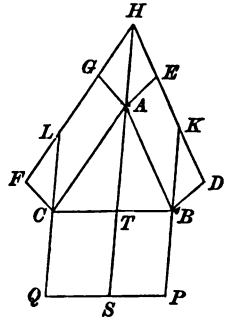
From (1) and (2), we have,

$$FC : QD :: FB : QE.$$

But, FC = FB, hence, QD = QE; the line FP must therefore pass through the middle of DE, and therefore, from Prop. LVI, Key, it must also pass through A, that is, the line through A and P is the third median of the given triangle. Hence, the three medians pass through P, *which was to be proved.*

PROP. LVIII.—*On the sides* AB *and* AC *of any triangle* ABC, *construct any two parallelograms* ABDE *and* ACFG; *prolong the sides* DE *and* FG *till they meet in* H; *draw* HA, *and on the third side of the triangle* BC, *construct a parallelogram two of whose sides are parallel and equal to* HA: *then show that the parallelogram on* BC *is equal to the sum of the parallelograms on* AB *and* AC.

DEMONSTRATION.—Let ABC be any triangle; Let AD and AF be any parallelograms constructed on AB and AC as sides; and let H be the point in which DE and FG meet, when prolonged; let CP be a parallelogram whose sides BP and CQ are parallel and equal to HA.



Draw HA, and prolong it to S; also prolong PB and QC, to K and L.

The parallelograms ABDE and ABKH have a common base AB, and a common altitude; hence, they are equal (Bk. IV, Prop. 1); the parallelograms ABKH and SPBT have equal bases, HA and ST, and a common altitude; they are therefore equal: hence, the parallelogram ABDE is equal to the parallelogram STBP. In like manner it may be shown that the parallelogram ACFG is equal to the parallelogram STCQ. The sum of the parallelograms TP and TQ is equal to the parallelogram BCQP; hence, the parallelogram BCQP is equal to the sum of the parallelograms ABDE and ACFG, *which was to be proved.*

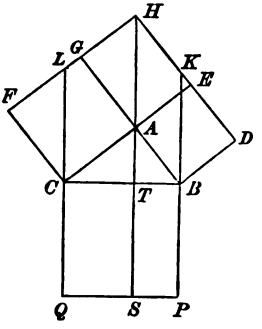
PROP. LIX.—*Assuming the principle demonstrated in the last proposition, deduce from it the truth that the square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.*

DEMONSTRATION.—Let ABC be a triangle right-angled at A, and let BQ, AD, and AF be squares described on its sides. Let DE and FG, when prolonged, meet in H.

Draw HA, and prolong it to S; prolong PB to K, and QC to L.

The right-angled triangle AGH has the side AG = AC, and the

side GH = AE = AB; hence, it is equal to the triangle CAB in all its parts, that is, AH is equal to CB, and the angle HAG is equal to the angle ACB. Since the angle CAG is a right angle, the sum of the angles HAG and CAT, or the sum of the angles ACB and CAT is equal to a right angle, and consequently the angle ATC is a right angle; hence, TS is parallel to CQ and BP. The square CP, is therefore a parallelogram whose sides are parallel and equal to HA,

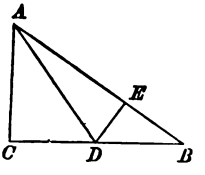


and consequently, from Prop. LVIII, Key, it is equal to the sum of the squares AD and AF, *which was to be proved.*

PROP. LX.—*If from the middle of the base of a right-angled triangle a line is drawn perpendicular to the hypothenuse dividing it into two segments, show that the difference of the squares of these segments is equal to the square of the other side about the right angle.*

DEMONSTRATION.—Let ACB be a right-angled triangle, and D the middle of its base.

Draw DE perpendicular to AB; draw also the line AD.



In the right-angled triangle ADE, we have,

$$\overline{AD}^2 = \overline{AE}^2 + \overline{ED}^2 \quad \ldots \quad (1).$$

In the right-angled triangle DEB, we have,

$$\overline{DB}^2, \text{ or } \overline{CD}^2 = \overline{EB}^2 + \overline{ED}^2 \quad \ldots \quad (2).$$

Subtracting (2) from (1), we have,

$$\overline{AD}^2 - \overline{CD}^2 = \overline{AE}^2 - \overline{EB}^2 \quad \ldots \quad (3).$$

But, $\overline{AD}^2 - \overline{CD}^2 = \overline{AC}^2$, hence, from (3), we have,

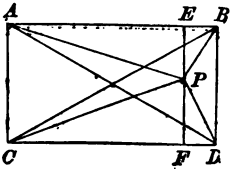
$$\overline{AE}^2 - \overline{EB}^2 = \overline{AC}^2,$$

which was to be proved.

PROP. LXI.—*If lines are drawn from any point* P *to the four vertices of a rectangle, show that the sum of the squares of the two lines drawn to the extremities of one diagonal is equal to the sum of the squares of the two lines drawn to the extremities of the other diagonal.*

DEMONSTRATION.—Let ABDC be any rectangle, and P any point in its plane.

Draw PA, PB, PC, and PD; also draw the diagonals AD and CB, and through P draw EPF perpendicular to CD.



From the right-angled triangles PEB and PFC, we have,

$$\overline{PB}^2 = \overline{PE}^2 + \overline{EB}^2$$

$$\overline{PC}^2 = \overline{PF}^2 + \overline{FC}^2,$$

whence, by addition,

$$\overline{PB}^2 + \overline{PC}^2 = \overline{PE}^2 + \overline{PF}^2 + \overline{EB}^2 + \overline{FC}^2 \ldots (1).$$

From the right-angled triangles PEA and PFD, we have,

$$\overline{PA}^2 = \overline{PE}^2 + \overline{EA}^2$$

$$\overline{PD}^2 = \overline{PF}^2 + \overline{FD}^2,$$

whence, by addition,

$$\overline{PA}^2 + \overline{PD}^2 = \overline{PE}^2 + \overline{PF}^2 + \overline{FD}^2 + \overline{EA}^2 \ldots (2).$$

Because, EB = FD, and EA = FC, the second members of (1) and (2) are equal; and consequently the first members are also equal; hence,

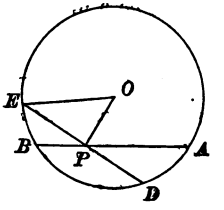
$$\overline{PB}^2 + \overline{PC}^2 = \overline{PA}^2 + \overline{PD}^2,$$

which was to be proved.

PROP. LXII.—*Let a line be drawn from the centre of a circle to any point of any chord; then show that the square of this line, plus the rectangle of the segments of the chord, is equal to the square of the radius.*

DEMONSTRATION.—Let O be the centre of any circle EBA, BA any chord, and P any point of that chord.

Draw OP, and through P draw the chord ED perpendicular to OP; also, draw EO.



From the right-angled triangle OPE, we have,

$$\overline{OE}^2 = \overline{OP}^2 + \overline{EP}^2 \ldots (1).$$

From Prop. 28, Bk. IV, we have,

$$EP \times PD, \text{ or } \overline{EP}^2 = BP \times PA;$$

substituting in (1), we have,

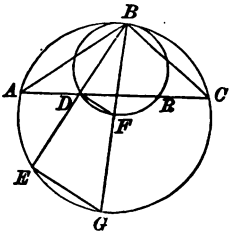
$$\overline{OE}^2 = \overline{OP}^2 + BP \times PA,$$

which was to be proved.

PROP. LXIII.—*Draw a line from the vertex of any scalene triangle to a point in the base, such that this line will be a mean proportional between the segments, into which it divides the base.*

SOLUTION.—Let ABC be a scalene triangle, AC its base, and B its vertex. Through A, B, and C pass a circle ABCG and let its centre

be F; draw the diameter BFG, and on the radius FB, as a diameter, describe a second circle cutting AC, in D and R; draw BE, DF and EG.



The angles BDF and BEG, being inscribed in semicircles, are right angles, and consequently, the triangles BDF and BEG are similar; hence, BD = DE.

From Prop. 28, Bk. IV, we have,

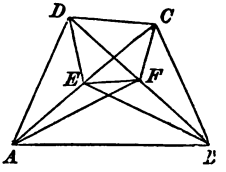
$$BD \times DE = AD \times DC, \text{ or } \overline{BD}^2 = AD \times DC;$$

hence, BD is the required line.

A second line could be found, by drawing a line from B to R, that would be a mean proportional between AR and RC.

PROP. LXIV.—*Show that the sum of the squares of the diagonals of any quadrilateral is equal to the sum of the squares of the four sides of the quadrilateral, increased by four times the square of the distance between the middle points of the diagonals.*

DEMONSTRATION.—Let ABCD be any quadrilateral, AC and BD its diagonals, and E and F the middle points of the diagonals.



Draw EF, ED, EB, FC and FA.

From Prop. 14, Bk. IV, we have the following relations:

$$\overline{CD}^2 + \overline{CB}^2 = 2\overline{BF}^2 + 2\overline{CF}^2$$

$$\overline{AD}^2 + \overline{AB}^2 = 2\overline{BF}^2 + 2\overline{AF}^2;$$

whence, by addition,

$$\overline{CD}^2 + \overline{CB}^2 + \overline{AD}^2 + \overline{AB}^2 = 4\overline{BF}^2 + 2(CF^2 + \overline{AF}^2).$$

But, from the same proposition, we have,

$$\overline{CF}^2 + \overline{AF}^2 = 2\overline{AE}^2 + 2\overline{EF}^2;$$

which gives, when substituted in the preceding equation,

$$\overline{CD}^2 + \overline{CB}^2 + \overline{AD}^2 + \overline{AB}^2 = 4\overline{BF}^2 + 4\overline{AE}^2 + 4\overline{EF}^2.$$

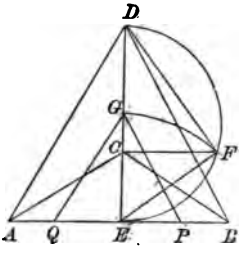
But, $4\overline{BF}^2 = \overline{BD}^2$, (Bk. IV, Prop. 8, Cor.); also $4\overline{AE}^2 = \overline{AC}^2$; hence,

$$\overline{CD}^2 + \overline{CB}^2 + \overline{AD}^2 + \overline{AB}^2 = \overline{BD}^2 + \overline{AC}^2 + 4\overline{EF}^2,$$

which was to be proved.

PROP. LXV.—*Construct an equilateral triangle equal in area to any isosceles triangle.*

SOLUTION.—Let ABC be any isosceles triangle. On AB, as a side, construct the equilateral triangle ABD; draw the line DCE, which is the common bisectrix of the angles D and C.



On DE, as a diameter, draw a semi-circle DFE, and from C erect the line CF perpendicular to DE; draw EF; and from E, as a centre, with EF as a radius, draw the arc FG, cutting DE in G; through G, draw GP and GQ, parallel to DB and DA. Also draw FD.

From Prop. 23, Bk. IV, Cor. 2, we have,

$$\overline{EF}^2, \text{ or } \overline{EG}^2 = EC \times ED, \text{ or}$$

$$ED : EG :: EG : EC \quad . \quad . \quad . \quad (1).$$

From the similar triangles EDB and EGP, we have,

$$ED : EG :: EB : EP \quad . \quad . \quad . \quad (2).$$

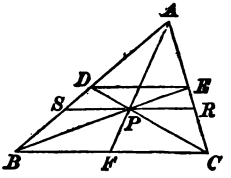
From (1) and (2), we have,

$$EG : EC :: EB : EP, \text{ or } EG \times EP = EC \times EB;$$

but, EG × EP is equal to the area of the equilateral triangle QGP, and EC × EB is equal to the area of the isosceles triangle ACB; hence, QGP is the required triangle.

PROP. LXVI.—*In a triangle* ABC *let two lines be drawn from the extremities of the base* BC, *intersecting at any point* P *on the median through* A, *and meeting the opposite sides in the points* E *and* D; *show that* DE *is parallel to* BC.

DEMONSTRATION.—Let ABC be the given triangle, P any point on the median, and CD, BE, lines drawn from C and B, through P.



Through P, draw SPR parallel to BC; also, join D and E.

From the similar triangles DSP and DBC, we have,

$$DP : DC :: SP : CB \quad . \quad . \quad . \quad (1).$$

From the similar triangles EPR and EBC, we have,

$$EP : EB :: PR, \text{ or } PS : BC \quad . \quad . \quad . \quad (2).$$

From (1) and (2), we have,

$$DP : DC :: EP : EB \quad . \quad . \quad . \quad (3).$$

Whence, by division (Bk. II, Prop. 6), we have,

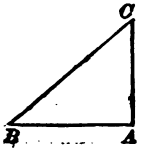
$$DP : DC - DP :: EP : EB - EP, \text{ or}$$
$$DP : PB :: EP : PC.$$

Hence, the triangles DPE and BPC are similar (Bk. IV, Prop. 20), and consequently, the angles DEP and PBC are equal; hence, DE is parallel to BC, *which was to be proved.*

APPLICATIONS OF ALGEBRA TO GEOMETRY.

PROP. LXVII.—*In a right-angled triangle* ABC, *given the base* BA, *and the sum of the hypothenuse and perpendicular to find the hypothenuse and the perpendicular.*

SOLUTION.—Denote BA by c, BC by x, AC by y, and the sum of BC and AC by s.



Then, $x + y = s \quad . \quad . \quad . \quad (1).$

From Bk. IV, Prop. 11, $x^2 = y^2 + c^2 \quad . \quad . \quad . \quad (2).$

From (1), we have, $x = s - y.$

Squaring, $x^2 = s^2 - 2sy + y^2 \quad . \quad . \quad . \quad (3).$

Subtracting (2) from (3), $0 = s^2 - 2sy - c^2.$

Transposing and dividing, $y = \frac{s^2 - c^2}{2s};$

whence, $x = s - \frac{s^2 - c^2}{2s} = \frac{s^2 + c^2}{2s}.$

If $c = 3$ and $s = 9$, we have $x = 5$ and $y = 4$.

PROP. LXVIII.—*In a right-angled triangle, given the hypothenuse and the sum of the sides about the right angle, to find these two sides.*

SOLUTION.—Denote BC by a, BA by x, CA by y, and BA + AC by s.

Then, $x + y = s \quad . \quad . \quad . \quad (1)$

and $x^2 + y^2 = a^2 \quad . \quad . \quad . \quad (2).$

From (1), $x^2 = s^2 - 2sy + y^2 \quad . \quad . \quad . \quad (3).$

Combining (2) and (3),

$$2y^2 - 2sy = a^2 - s^2 \quad . \quad . \quad . \quad (4),$$

whence,

$$y^2 - sy = \frac{a^2 - s^2}{2} \quad . \quad . \quad . \quad (5).$$

Solving (5),

$$y = \tfrac{1}{2}s \pm \sqrt{\tfrac{1}{2}a^2 - \tfrac{1}{4}s^2} \quad . \quad . \quad . \quad (6).$$

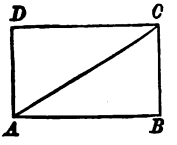
Substituting in (1),

$$x = \tfrac{1}{2}s \mp \sqrt{\tfrac{1}{2}a^2 - \tfrac{1}{4}s^2} \quad . \quad . \quad . \quad (7).$$

If $a = 5$ and $s = 7$, we have, $y = 4$ and 3, $x = 3$ and 4.

PROP. LXIX.—*In a rectangle, given the diagonal and the perimeter, to find the sides.*

SOLUTION.—Denote AC by d, the side BA by x, the side BC by y, and the sum of BA and BC, or half the perimeter by a.



Then,

$$x + y = a,$$

and,

$$x^2 + y^2 = d^2,$$

from which we obtain by solution,

$$y = \tfrac{1}{2}a \pm \sqrt{\tfrac{1}{2}d^2 - \tfrac{1}{4}a^2}$$

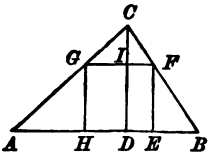
$$x = \tfrac{1}{2}a \mp \sqrt{\tfrac{1}{2}d^2 - \tfrac{1}{4}a^2}.$$

If $d = 10$ and $a = 14$, we have $y = 8$ and 6, and $x = 6$ and 8.

PROP. LXX.—*Given the base and perpendicular of a triangle, to find the side of an inscribed square.*

SOLUTION.—Let ABC be the given triangle, and let FGHE be the inscribed square.

Denote AB by b, CD by a, GH by x, whence CI is equal to $a - x$.



From the similar triangles ACB and GCF,

we have, $\quad AB : CD :: GF : CI,$

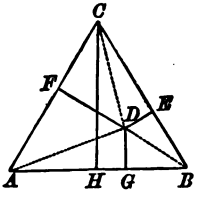
or, $\quad b : a :: x : a - x,$

whence, $\quad x = \dfrac{ab}{a + b}.$

The side of the inscribed square depends only on the base and altitude of the given triangle.

PROP. LXXI.—*In an equilateral triangle, given the distances from a point within to the three sides, to find one side of the triangle.*

SOLUTION.—Let ACB be an equilateral triangle, D any point within it, DE, DG, and DF perpendiculars from D to the sides of the triangle and CH the altitude of the triangle.



Draw DA, DB, and BC.

Denote DG by a, DE by b, and DF by c; also denote one of the sides of the triangle by $2x$, hence, AH $= x$ and CH $= \sqrt{\overline{AC}^2 - \overline{AH}^2} = \sqrt{4x^2 - x^2}$, or CH $= x\sqrt{3}$.

The area of the given triangle is equal to $x \times x\sqrt{3}$, or $x^2\sqrt{3}$;

the area of the triangle ADB is equal to $a \times x$;

the area of the triangle BDC is equal to $b \times x$;

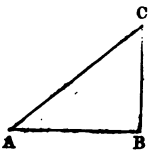
the area of the triangle ADC is equal to $c \times x$.

But the last three triangles make up the first; hence,

$$x^2\sqrt{3} = ax + bx + cx,$$

whence, by solution, $\quad x = \dfrac{a + b + c}{\sqrt{3}}.$

PROP. LXXII.—*In a right-angled triangle, having given the base and difference between the hypothenuse and perpendicular, to find the sides.*



Let a = the base AB,

b = the difference between AC and BC,

x = the hypothenuse AC, and

y = the perpendicular CB.

Then, $x - y = b$, by the conditions, ... (1)

$x^2 = a^2 + y^2$ (Bk. IV. Prop. 11), . . . (2)

From equation (1) we have,

$$x = y + b;$$

by squaring, $$x^2 = y^2 + 2by + b^2.$$

Substituting this value of x^2 in equation (2),

$$y^2 + 2by + b^2 = a^2 + y^2;$$

hence, $2by = a^2 - b^2$, or, $y = \frac{a^2 - b^2}{2b}$;

substituting this value in equation (1), we readily find

$$x = \frac{a^2 + b^2}{2b}$$

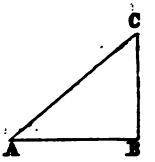
PROP. LXXIII.—*In a right-angled triangle, having given the hypothenuse and the difference between the base and perpendicular, to determine the triangle.*

Let a = the hypothenuse AC,

b = difference between AB and BC,

x = the base AB, and

y = the perpendicular BC.



Then, $x - y = b$, by the conditions, . . (1)

$a^2 = x^2 + y^2$, (Bk. IV, Prop. 11) . (2)

From equation (1), we have,

$$x = y + b;$$ and by squaring,

$$x^2 = y^2 + 2by + b^2.$$

Substituting this value of x^2 in equation (2), we have,

$$a^2 = y^2 + 2by + b^2 + y^2;$$

and by transposing and reducing,

$$y^2 + by = \frac{a^2 - b^2}{2};$$

hence, $$y = \frac{-b + \sqrt{2a^2 - b^2}}{2}; \qquad y = -\left(\frac{b + \sqrt{2a^2 - b^2}}{2}\right);$$

Substituting either of these values in equation (1), we readily find the corresponding value of x.

NOTE.—The *positive value* of the unknown quantity *generally* fulfils the conditions of the problem, understood in its arithmetical sense.

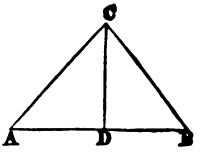
The negative value will always satisfy the *conditions of the equation:* with its sign changed, it may be regarded as the answer to a problem which differs from the one proposed only in this: *that certain quantities which were additive have become subtractive, and the reverse.*

PROP. LXXIV.—*Having given the area of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.*

By a given triangle, we mean one whose sides are all known, or given.

1st. *To find the perpendicular and segments of the base:*

Let ABC be a triangle, in which the three sides are given: viz.,



$b = \text{AB}$, $a = \text{AC}$, and $c = \text{BC}$.

Let CD be drawn perpendicular to AB, and let $y = \text{CD}$, and $x = \text{AD}$; then will $b - x = \text{DB}$.

Then, $$a^2 = x^2 + y^2 \quad \cdots\cdots\cdots \quad (1)$$

$$c^2 = y^2 + x^2 - 2bx + b^2 \quad \cdots \quad (2);$$

substituting in equation (2) the value of $x^2 + y^2 = a^2$, from equation (1),

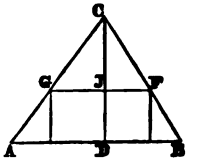
$$c^2 = a^2 - 2bx + b^2; \quad \text{hence,}$$

$$x = \frac{a^2 + b^2 - c^2}{2b};$$

and substituting this value in equation (1), we find the altitude of the triangle.

2d. *To find the sides of the rectangle.*

Suppose the rectangle to be inscribed in the triangle ACB.



Let $d =$ the area of the rectangle,

$x =$ its base,

$y =$ its altitude, and

$h = \text{CD}$, the altitude of the triangle.

Then, by similar triangles,

$$\text{AB} : \text{CD} :: \text{GF} : \text{CJ}; \quad \text{that is,}$$

$$b : h :: x : h - y; \quad \text{hence,}$$

$$hx = bh - by \quad \cdots \quad (1),$$

and $$xy = d, \quad \text{by the conditions, (2).}$$

From equation (2), we have $$x = \frac{d}{y};$$

substituting this value in equation (1), we have,

$$h\frac{d}{y} = bh - by\,; \quad \text{clearing of fractions,}$$

$$hd = bhy - by^2\,; \quad \text{or,}$$

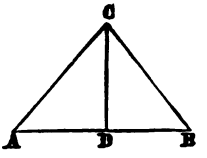
$$y^2 - hy = \frac{-hd}{b}\,; \quad \text{whence,}$$

$$y = +\frac{h}{2} + \frac{1}{2}\sqrt{\frac{b\,h^2 - 4hd}{b}}\,; \qquad y = +\frac{h}{2} - \frac{1}{2}\sqrt{\frac{b\,h^2 - 4hd}{b}}\,.$$

Substituting these values of y in equation (1), we find the corresponding values of x.

PROP. LXXV.—*In a triangle, having given the ratio of the two sides, together with both the segments of the base made by a perpendicular from the vertical angle, to determine the triangle.*

Let ACB be a triangle, and CD a line drawn perpendicular to the base AB.



Let $AC = x$

$CB = y$

$CD = z$

$AD = a$

$DB = b$

$c =$ ratio.

Then, $\frac{x}{y} = c$, by the conditions . . . (1)

$x^2 - a^2 = z^2$ (Bk. IV., Prop. 11, Cor. 1) . . (2)

$y^2 - b^2 = z^2$ (Bk. IV., Prop. 11, Cor. 1) . . (3).

Subtracting (3) from (2), member from member,

$$x^2 - y^2 + b^2 - a^2 = 0\,; \quad \text{or,} \quad x^2 = y^2 + a^2 - b^2 \quad . \quad (4).$$

From equation (1) we have

$$x = cy; \quad \text{hence,} \quad x^2 = c^2 y^2 \quad \cdots \quad (5);$$

combining (4) and (5), we have

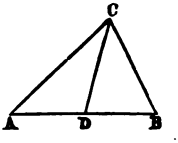
$$c^2y^2 = y^2 + a^2 - b^2, \quad \text{and} \quad (c^2 - 1)\, y^2 = a^2 - b^2;$$

hence, $$y^2 = \frac{a^2 - b^2}{c^2 - 1}; \quad \text{and} \quad y = \pm\sqrt{\frac{a^2 - b^2}{c^2 - 1}};$$

substituting these values in equation (1), we find the corresponding values of x.

PROP. LXXVI.—*In a triangle, having given the base, the sum of the other two sides and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.*

Let ABC be a triangle, and CD the line drawn from the vertex C to D, the middle point of the base.



Let
$$AC = x$$
$$BC = y$$
$$AD = b$$
$$CD = a$$
$$AC + BC = s.$$

Then, $$x + y = s, \quad \text{by the conditions} \quad \cdots \quad (1)$$

and $$x^2 + y^2 = 2b^2 + 2a^2, \text{ (Bk. IV., Prop. 14)} \quad \cdots \quad (2);$$

equation (1), by transposing and squaring, gives

$$y^2 = x^2 - 2sx + s^2;$$

substituting this value in equation (2), we have

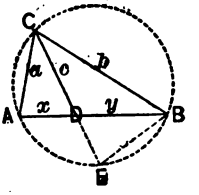
$$2x^2 - 2sx + s^2 = 2b^2 + 2a^2;$$

transposing and reducing, we have

$$x^2 - sx = \frac{2b^2 + 2a^2 - s^2}{2}: \text{ whence,}$$

$$x = \frac{s + \sqrt{4b^2 + 4a^2 - s^2}}{2}, \text{ and } x = \frac{s - \sqrt{4b^2 + 4a^2 - s^2}}{2}.$$

PROP. LXXVII.—*In a triangle, having given the two sides about the vertical angle, together with a line bisecting that angle and terminating in the base, to find the base.*



Circumscribe a circle about the triangle and complete construction as in figure. CEB = CAD (B. III, P. 18); ACD = ECB by hypothesis; hence triangles ACD and CEB are similar (B. IV, P. 18, C.); hence, AC : CE :: CD : CB,

or a : CE :: c : b; $\therefore$ CE $= \frac{ab}{c}$. DE $=$ CE $-$ CD $= \frac{ab}{c} - c$: but CD $\times$ DE $=$ AD $\times$ DB (B. IV, P. 28, C.); that is, $c\left(\frac{ab}{c} - c\right) = xy$; $\therefore$ $ab - c^2 = xy$, or $ab = xy + c^2$. (1)

Again, AD : DB :: AC : CB (B. IV, P. 17), or $x : y :: a : b$;

$$\therefore bx = ay. \quad (2)$$

Finding x and y from (1) and (2), and adding, we have

$$x + y, \text{ or AB} = \sqrt{\frac{a(ab - c^2)}{b}} + \sqrt{\frac{b(ab - c^2)}{a}}.$$

PROP. LXXVIII.—*To determine a right-angled triangle, having given the lengths of two lines drawn from the vertices of the acute angles to the middle points of the opposite sides.*

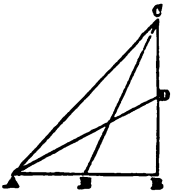
Let ABC be a right angled triangle, and AD, CE, two lines drawn to the middle points D and E of the opposite sides.

Let AD $= a$

CE $= b$

AB $= 2x$

BC $= 2y$



Then, $4x^2 + y^2 = a^2$ (Bk. IV. Prop. 11) . (1)

and, $4y^2 + x^2 = b^2$ " " " " . . (2).

Multiplying equation (1) by 4, and subtracting (2) from it, we have,

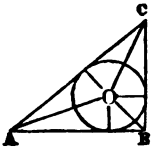
$$15x^2 = 4a^2 - b^2; \qquad \text{when, } x = \pm\sqrt{\frac{4a^2 - b^2}{15}};$$

and substituting in equation (2) $\quad y = \pm\sqrt{\frac{4b^2 - a^2}{15}}.$

PROP. LXXIX.—*To determine a right-angled triangle, having given the perimeter and the radius of the inscribed circle.*

1st. Let ABC be a right angled triangle, O the centre, and r the radius of the inscribed circle.

Let p = the perimeter; let x denote the length of the equal tangents drawn from A (Bk. III. Prop. 14, sch.); v the length of the equal tangents drawn from B; and y the length of the equal tangents drawn from C.



Then, AC $= x + y$, AB $= x + v$, and BC $= y + v$; then,

$$\text{AC} + \text{AB} + \text{BC} = 2x + 2y + 2v = p;$$

transposing and reducing,

$$x + y = \frac{p - 2v}{2} = a, \text{ a known quantity} \quad . \quad . \quad . \quad (1).$$

Then, $AB^2 + BC^2 = AC^2$; that is,

$$x^2 + 2vx + v^2 + y^2 + 2vy + v^2 = a^2 \quad . \quad . \quad . \quad . \quad (2).$$

2nd. Observe that the double area of each of the triangles AOB, BOC, and AOC, is equal to its base multiplied by the radius of the inscribed circle; and hence, the sum of these products is equal to the sum of the bases multiplied by r; that is, $= r \times p$, a known quantity.

But the base AB × BC is also equal to double the area of the triangle ABC; hence,

$$(x + v) \times (y + v) = r \times p; \text{ that is,}$$

$$xy + vx + vy + v^2 = r \times p \quad . \quad . \quad . \quad . \quad (3);$$

Multiplying both numbers of equation (3) by 2, we have,

$$2xy + 2vx + 2vy + 2v^2 = 2rp \quad . \quad . \quad . \quad (4);$$

subtracting equation (4) from (2), we have,

$$x^2 - 2xy + y^2 = a^2 - 2rp; \text{ whence, by extracting square root,}$$

$$x - y = \pm \sqrt{a^2 - 2rp} = b \quad . \quad . \quad . \quad . \quad . \quad . \quad (5);$$

combining (1) and (5) we readily find the values of x and y,

$$x = \frac{a + b}{2}, \quad \text{and} \quad y = \frac{a - b}{2}.$$

PROP. LXXX.—*To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.*

Let ACB be a triangle, and CD perpendicular to the base AB.

Let $AC = y$,

r = ratio;

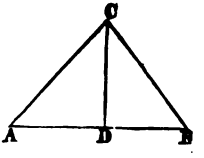
then $ry = CB$

$AB = b$

$DB = b - x$

$AD = x$

$CD = h$.



Then, $y^2 = x^2 + h^2$; and $r^2y^2 = b^2 - 2bx + x^2 + h^2$.

Multiplying the first equation by r^2, and subtracting,

$$0 = b^2 - 2bx + x^2 + h^2 - r^2x^2 - r^2h^2; \text{ whence,}$$

$$(1 - r^2)\, x^2 - 2bx = (r^2 - 1)\, h^2 - b^2; \text{ or,}$$

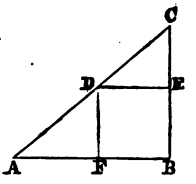
$$x^2 - \frac{2b}{1 - r^2} \cdot x = \frac{(r^2 - 1)\, h^2 - b^2}{1 - r^2}; \text{ whence,}$$

$$x = \frac{b \pm \sqrt{[(r^2 - 1)\, h^2 - b^2]\,(1 - r^2) + b^2}}{1 - r^2}.$$

PROP. LXXXI.—*To determine a right-angled triangle, having given the hypothenuse, and the side of the inscribed square.*

Let ACB be a right-angled triangle, and FDEB an inscribed square.

Let $AC = h$, and $DF = s$: also, denote AB by x, and BC by y; then $CE = y - s$.



Then, AB : BC :: DE : EC;

that is, $x : y :: s : y - s$;

whence, $xy - sx = sy$; or,

$$xy = sy + sx = s\,(x + y) \quad \cdot \cdot \quad (1);$$

also, $x^2 + y^2 = h^2$ (Bk. IV., Prop. 11) $\cdot \cdot$ (2).

If to equation 2, we add twice equation (1), we have

$$x^2 + 2xy + y^2 = 2s\,(x + y) + h^2; \text{ or,}$$

$$(x + y)^2 - 2s\,(x + y) = h^2 \quad \cdot \cdot \quad (3),$$

which is an equation of the second degree, in which the unknown quantity is $x + y$; hence,

$$x + y = s + \sqrt{h^2 + s^2} \quad \cdot \cdot \quad (4).$$

Combining equations (1) and (4), we have

$$xy = s^2 + s\sqrt{h^2 + s^2} \quad \cdot \cdot \quad (5).$$

From the square of equation (4), subtract 4 times equation (5), and we have

$$x^2 - 2xy + y^2 = h^2 - 2s^2 - 2s\sqrt{h^2 + s^2} \quad \cdot \cdot \quad (6);$$

extracting the square root of both members,

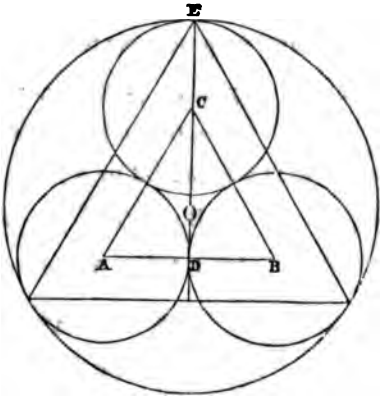
$$x - y = \sqrt{h^2 - 2s^2 - 2s\sqrt{h^2 + s^2}} \quad \cdot \cdot \quad (7),$$

combining equations (4) and (7), we have

$$x = \frac{s + \sqrt{h^2 + s^2} + \sqrt{h^2 - 2s^2 - 2s\sqrt{h^2 + s^2}}}{2} \quad \cdot \cdot \quad (8),$$

$$y = \frac{s + \sqrt{h^2 + s^2} - \sqrt{h^2 - 2s^2 - 2s\sqrt{h^2 + s^2}}}{2} \quad \cdot \quad (9).$$

PROP. LXXXII.—*To determine the radii of three equal circles, described within and tangent to, a given circle, and also tangent to each other.*



Let O be the centre of the given circle, and A, B and C, the centres of the equal inscribed circles.

Denote the radius of the given circle by R, and the equal radii of the inscribed circles by r.

Joining the centres A and C, C and B, B and A, by straight lines, we have the equilateral triangle ABC, each of whose sides is $2r$. Draw COD and prolong it to E, and it will be perpendicular to AB.

Then, in the right-angled triangle ACD, we have

$$CD^2 = AC^2 - AD^2, \quad \text{or}$$

$$CD^2 = 4r^2 - r^2 = 3r^2, \quad \text{or} \quad CD = r\sqrt{3}.$$

But since O is the point at which lines drawn from the vertices of the angles to the middle points of the opposite sides, in both triangles, intersect each other, it follows, from Cor. of Prob. 21, that

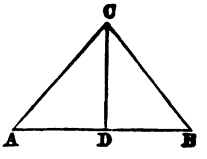
$$CO = \tfrac{2}{3}\,CD = \tfrac{2}{3}\,r\sqrt{3}; \quad \text{hence,}$$

$$OE = R = r + \tfrac{2}{3}\,r\sqrt{3}; \quad \text{hence, finally,}$$

$$r = \frac{3R}{3 + 2\sqrt{3}} = \frac{R}{1 + 2\sqrt{\tfrac{1}{3}}}.$$

PROP. LXXXIII.—*In a right-angled triangle, having given the perimeter and the perpendicular let fall from the right angle on the hypothenuse, to determine the triangle.*

Let ACB be a right-angled triangle, right angled at C; and let CD be drawn perpendicular to the hypothenuse AB.



Let p = the perimeter, and

h = the perpendicular CD.

Denote AC by x, and CB by y;

then, $AB = \sqrt{x^2 + y^2}$, and

$AC + CB + AB = p$; that is,

$$x + y + \sqrt{x^2 + y^2} = p \quad \cdot \quad (1).$$

Again, xy = double the area of ACB $\cdot \cdot$ (2),

and $h\sqrt{x^2 + y^2}$ = double area $\cdot \cdot$ (3);

hence, $$xy = h\sqrt{x^2 + y^2} \quad \cdot \cdot \quad (4)$$

transposing in equation (1) and squaring,

$$(x+y)^2 = (p - \sqrt{x^2+y^2})^2,$$

$$x^2 + 2xy + y^2 = p^2 - 2p\sqrt{x^2+y^2} + x^2 + y^2,$$

or $$2xy = p^2 - 2p\sqrt{x^2+y^2} \quad \cdots \quad (5);$$

multiplying both members of equation (4) by 2,

$$2xy = 2h\sqrt{x^2+y^2} \quad \cdots \quad (6);$$

combining equations (5) and (6),

$$2h\sqrt{x^2+y^2} = p^2 - 2p\sqrt{x^2+y^2}; \text{ hence,}$$

$$\sqrt{x^2+y^2} = \frac{p^2}{2(h+p)} \quad \cdots \quad (7);$$

substituting this value of $\sqrt{x^2+y^2}$, in equation (4),

$$xy = \frac{hp^2}{2(h+p)} \quad \cdots \quad (8):$$

squaring both members of equation (7), we have

$$x^2 + y^2 = \frac{p^4}{4(h+p)^2};$$

adding and subtracting 2 times each member of (8),

$$x^2 + 2xy + y^2 = \frac{p^2(p^2+4ph+4h^2)}{4(h+p)^2} \quad \cdots \quad (9),$$

$$x^2 - 2xy + y^2 = \frac{p^2(p^2-4ph-4h^2)}{4(h+p)^2} \quad \cdots \quad (10).$$

Extracting the square roots of equations (9) and (10),

$$x + y = \frac{p(p+2h)}{2(h+p)} \quad \cdots \quad (11),$$

$$x - y = \frac{p\sqrt{p^2-4ph-4h^2}}{2(h+p)} \quad \cdots \quad (12);$$

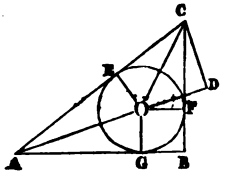
hence, $$x = \frac{p(p+2h) + p\sqrt{p^2 - 4ph - 4h^2}}{4(h+p)} \quad . \quad . \quad (13),$$

$$y = \frac{p(p+2h) - p\sqrt{p^2 - 4ph - 4h^2}}{4(h+p)} \quad . \quad . \quad (14).$$

PROP. LXXXIV.—*To determine a right-angled triangle, having given the hypothenuse and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.*

Let ABC be a right-angled triangle, right-angled at B, and AO, OC, two lines drawn from the acute angles to the centre O of the inscribed circle.

Let $AC = h$; $AO - OC = d$, $OC = x$;
then, $AO = x + d$



Produce AO, and from C draw CD perpendicular to the prolongation, meeting it at D. Then, since the sum of the angles BAC and ACB is equal to a right-angle (Bk. I, Prop. 25, Cor. 4), and since the lines AO and CO bisect these angles (Bk. III, Prob. 14), OAC + ACO is equal to a half a right-angle.

Since the outward angle COD is equal to the sum of the inward angles (Bk. I, Prop. 25, Cor. 6), it is equal to half a right-angle; and hence, OCD is equal to half a right-angle, and hence OD and CD are equal. Denote either by z; then,

$$x^2 = z^2 + z^2; \quad \text{and} \quad z = x\sqrt{\tfrac{1}{2}}$$

Then, $AC^2 = AD^2 + CD^2$; that is,

$$h^2 = (x + d + x\sqrt{\tfrac{1}{2}})^2 + (x\sqrt{\tfrac{1}{2}})^2;$$

and by reduction, $(2+\sqrt{2})x^2 + d(2+\sqrt{2})x = h^2 - d^2$; or,

$$x^2 + dx = \frac{h^2 - d^2}{2+\sqrt{2}}$$

hence, $$x = -\tfrac{1}{2}d + \tfrac{1}{2}\sqrt{\frac{4h^2 - (2-\sqrt{2})\,d^2}{2+\sqrt{2}}},$$

$$x = -\tfrac{1}{2}d - \tfrac{1}{2}\sqrt{\frac{4h^2 - (2-\sqrt{2})\,d^2}{2+\sqrt{2}}}.$$

Let $OC = x = m$, $AO = m + d$, $CD = z = m\sqrt{\frac{1}{2}}$,

$AB = y$, and $CB = u$.

Then, since the triangles ACD and AOG are similar,

$$h : m\sqrt{\tfrac{1}{2}} :: m + d : OG = r = \frac{m\sqrt{\frac{1}{2}} \times (m+d)}{h}; \text{ and}$$

$$h : m + d + m\sqrt{\tfrac{1}{2}} :: m + d : AG = \frac{(m+d)\,(m+d+m\sqrt{\frac{1}{2}})}{h}$$

$$AB = y = AG + GB = \frac{(m+d)\,(m+d+m\sqrt{\frac{1}{2}}) + m\sqrt{\frac{1}{2}}(m+d)}{h}$$

$$= \frac{(m+d)\,(m+d+m\sqrt{2})}{h} = b; \quad \text{and}$$

$$u = \sqrt{h^2 - b^2}.$$

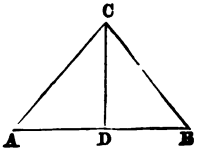
PROP. LXXXV.—*To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.*

Let ACB be a triangle, and CD perpendicular to the base AB

Let $AB = b$, $CD = a$, $AC - BC = d$,

$BC = x$, $AC = x + d$, and $DB = z$,

and $AD = b - z$.



Then, $AC = \sqrt{a^2 + (b-z)^2}$;

and $CB = \sqrt{a^2 + z^2}$; hence,

$$\sqrt{a^2 + (b-z)^2} - \sqrt{a^2 + z^2} = d.$$

Transposing and squaring, we have

$$a^2 + (b-z)^2 = d^2 + 2d\sqrt{a^2 + z^2} + a^2 + z^2;$$

squaring $b - z$, in the first member, we have

$$a^2 + b^2 - 2bz + z^2 = d^2 + 2d\sqrt{a^2 + z^2} + a^2 + z^2;$$

whence, by reducing,

$$(b^2 - d^2) - 2bz = 2d\sqrt{a^2 + z^2};$$

squaring both members, we have

$$(b^2 - d^2)^2 - 4b(b^2 - d^2).z + 4b^2z^2 = 4a^2d^2 + 4d^2z^2;$$

transposing, and collecting the terms,

$$4(b^2 - d^2).z^2 - 4b(b^2 - d^2).z = 4a^2d^2 - (b^2 - d^2)^2; \quad \text{or,}$$

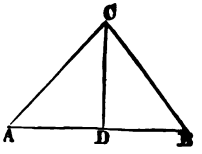
$$z^2 - bz = \frac{4a^2d^2 - (b^2 - d^2)^2}{4(b^2 - d^2)}; \quad \text{whence,}$$

$$z = \frac{b}{2} \pm \frac{1}{2}\sqrt{\frac{4a^2d^2 + 2b^2d^2 - d^4}{b^2 - d^2}};$$

from which the sides AC and CB are easily found.

PROP. LXXXVI.—*To determine a triangle, having given the base, the perpendicular, and the rectangle of the other sides.*

Let ACB be a triangle, and CD a line drawn perpendicular to the base AB.



Let $AB = b$, $CD = d$, $AC \times CB = q$;

$AC = x$, and $BC = y$; then,

$$xy = q \quad . \quad . \quad . \quad (1).$$

$AD = \sqrt{x^2 - d^2}$, and $DB = \sqrt{y^2 - d^2}$; hence,

$$\sqrt{x^2 - d^2} + \sqrt{y^2 - d^2} = b \quad . \quad . \quad . \quad (2).$$

Transposing, and then squaring both members of equation (2), we have

$$x^2 - d^2 = b^2 - 2b\sqrt{y^2 - d^2} + y^2 - d^2;$$

whence, $(x^2 - y^2) - b^2 = -2b\sqrt{y^2 - d^2}$; squaring again,

$x^4 - 2x^2y^2 + y^4 - 2b^2(x^2 - y^2) + b^4 = 4b^2y^2 - 4b^2d^2$; reducing,

$$x^4 - 2x^2y^2 + y^4 - 2b^2(x^2 + y^2) = -b^4 - 4b^2d^2 \quad \cdot \cdot \quad (3).$$

Adding 4 times the square of each member of (1),

$$x^4 + 2x^2y^2 + y^2 - 2b^2(x^2 + y^2) = 4q^2 - b^4 - 4b^2d^2; \text{ or,}$$

$$(x^2 + y^2)^2 - 2b^2(x^2 + y^2) = 4q^2 - b^4 - 4b^2d^2.$$

Regarding $x^2 + y^2$ as a single unknown quantity, we have

$$x^2 + y^2 = b^2 \pm \sqrt{4q^2 - 4b^2d^2} \quad \cdot \cdot \cdot \quad (4);$$

then, by adding twice each member of equation (1),

$$x^2 + 2xy + x^2 = 2q + b^2 \pm \sqrt{4q^2 - 4b^2d^2}; \text{ or,}$$

$$x + y = \sqrt{2q + b^2 \pm \sqrt{4q^2 - 4b^2d^2}} = m;$$

subtracting from equation (4), twice equation (1),

$$x^2 - 2xy + x^2 = -2q + b^2 \pm \sqrt{4q^2 - 4b^2d^2}; \text{ or,}$$

$$x - y = \pm\sqrt{-2q + b^2 \pm \sqrt{4q^2 - 4b^2d^2}} = n;$$

hence, $$x = \frac{m+n}{2}, \text{ and } y = \frac{m-n}{2}.$$

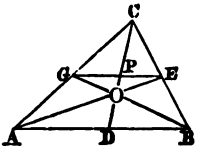
PROP. LXXXVII.—*To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.*

Let ACB be a triangle, and AE, BG, CD, three lines drawn from the vertices to the middle points of the opposite sides.

Let CD$=a$, AE$=b$, BG$=c$, AC$=x$, BC$=y$, and AB$=z$: then,

$$x^2 + y^2 = 2a^2 + \frac{z^2}{2} \text{ (Bk. IV. Prop. 14),}$$

$$x^2 + z^2 = 2b^2 + \frac{y^2}{2}; \quad y^2 + z^2 = 2c^2 + \frac{x^2}{2};$$

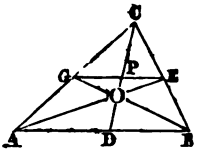


clearing of fractions and transposing, we have

$$2x^2 + 2y^2 - z^2 = 4a^2 \quad \cdots \quad (1)$$

$$2x^2 + 2z^2 - y^2 = 4b^2 \quad \cdots \quad (2)$$

$$2y^2 + 2z^2 - x^2 = 4c^2 \quad \cdots \quad (3);$$



subtracting equation (2) from (1), we have

$$3y^2 - 3z^2 = 4(a^2 - b^2) \quad \cdots \quad (4);$$

multiplying equation (3) by 2, and adding to (2),

$$3y^2 + 6z^2 = 8c^2 + 4b^2 \quad \cdots \quad (5);$$

subtracting equation (4) from (5),

$$9z^2 = 8c^2 + 4b^2 - 4a^2 + 4b^2; \quad \text{hence,}$$

$$z = \pm \tfrac{2}{3}\sqrt{2b^2 + 2c^2 - a^2}$$

$$x = \pm \tfrac{2}{3}\sqrt{2a^2 + 2b^2 - c^2}$$

$$y = \pm \tfrac{2}{3}\sqrt{2a^2 + 2c^2 - b^2}.$$

Cor. Through G and E, the middle points of AC and BC, draw GE; then will GE be parallel to the base AB (Bk. IV. Prop. 16): and since AC is equal to twice CG, AD will be equal to twice GP, or its equal PE.

But the triangles AOD and POE are similar; and, since AD is equal to twice PE, AO is equal to twice OE; that is,

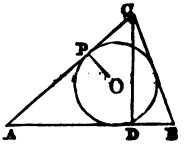
If three lines be drawn from the vertices of the three angles of a triangle to the middle points of the opposite sides, the distance from either vertex to the point of intersection, will be two-thirds of the bisecting line.

PROP. LXXXVIII.—*In a triangle, having given the three sides, to find the radius of the inscribed circle.*

Let ABC be a triangle, CD perpendicular to the base, O the centre, and OP the radius of the inscribed circle.

Let $BC = a$, $AB = c$, $AC = b$,

$OP = r$, $AD = z$, $CD = x$.



$x^2 = b^2 - z^2$ . . . (1),

$x^2 = a^2 - (c - z)^2$; whence,

$b^2 - z^2 = a^2 - c^2 + 2cz - z^2$; therefore,

$$z = \frac{b^2 + c^2 - a^2}{2c};$$

combining with equation (1),

$$x = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c};$$

area $= (a + b + c)\,\frac{1}{2}r = \frac{1}{2}cx$; whence,

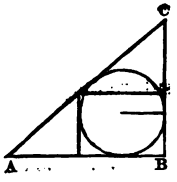
$$r = \frac{cx}{a + b + c} = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2(a + b + c)}.$$

PROP. LXXXIX.—*To determine a right-angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.*

Let ABC be a right-angled triangle, with a square and circle both inscribed.

Let $AB = x$, $BC = y$, and $AC = z$:

denote the side of the square by s, and the radius of the circle by r.



Then, $x + y - z = 2r$ (Prop. LXXIX, Key) . (1),

$x^2 + y^2 = z^2$ . . . . . (2),

and $x + y = s + \sqrt{s^2 + s^2}$ (Prop. LXXXI, Key) . . (3).

Combining equations (1) and (3), and transposing,

$$z + 2r - s = \sqrt{s^2 + s^2};\ \text{squaring},$$

$$z^2 + 4r^2 + s^2 + 4rz - 2sz - 4rs = z^2 + s^2;$$

whence, $$z = \frac{2rs - 2r^2}{2r - s} \quad . \quad . \quad . \quad (4);$$

combining equations (4) and (1),

$$x + y = \frac{2r^2}{2r - s} = m, \text{ and equation (2) gives}$$

$$x^2 + y^2 = \frac{(2rs - 2r^2)^2}{(2r - s)^2} = n^2; \text{ whence,}$$

$$x = \tfrac{1}{2}(m + \sqrt{2n^2 - m^2}),$$

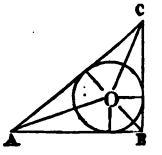
$$y = \tfrac{1}{2}(m - \sqrt{2n^2 - m^2}).$$

PROP. XC.—*To determine a right-angled triangle, having given the hypothenuse and radius of the inscribed circle.*

Let ABC be a right-angled triangle, and O the centre of the inscribed circle.

Let $AC = h$, $AB = x$, $BC = y$,

and $r =$ the radius of the circle.



$x + y = h + 2r$ (Prop. LXXIX, Key) . (1)

$x^2 + y^2 = h^2$ . . . . . (2).

Since the perimeter is equal to $2h + 2r$, and since four times the area is equal to the perimeter into $2r$,

$$2xy = 4r^2 + 4hr \quad . \quad . \quad . \quad (3);$$

subtracting equation (3) from (2), and extracting the square root,

$$x - y = \sqrt{h^2 - 4r^2 - 4rh} \quad . \quad . \quad . \quad (4);$$

combining (4) and (1),

$$x = \frac{h + 2r + \sqrt{h^2 - 4r^2 - 4rh}}{2},$$

$$y = \frac{h + 2r - \sqrt{h^2 - 4r^2 - 4rh}}{2}.$$

SUPPLEMENT.

A KEY

TO THE

EXERCISES

APPENDED TO THE SEVERAL BOOKS OF

DAVIES' LEGENDRE.

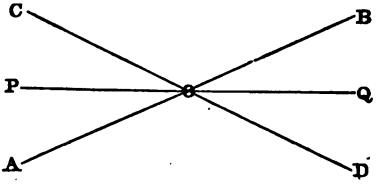
NEW YORK .:. CINCINNATI .:. CHICAGO
AMERICAN BOOK COMPANY

KEY.

BOOK I.

1.—*Show that the lines which bisect* (halve) *two vertical angles, form one and the same straight line.*

Let COA and BOD be two vertical angles, and let OP and OQ be the lines which bisect them, then will OP and OQ form one and the same straight line.



Since (P. II.) COA = BOD, POA = BOQ (A, 7); hence COA = COP + BOQ: but COA + COB = 2 right angles (P. I.), and substituting for COA, its equal, COP + BOQ, we have COP + BOQ + COB = 2 right angles or, since COP + COB = BOP, we have BOP + BOQ = 2 right angles; hence

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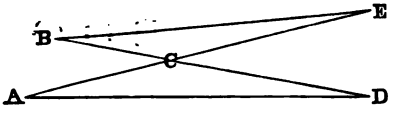
NOTE.—In making references, the following abbreviations are employed, viz.: A. for Axiom; B. for Book; C. for Corollary; D. for Definition; E. for Exercise; P. for Proposition; Prob. for Problem; and S. for Scholium. In referring to the same Book of the Legendre in which the exercise occurs, the number of the Book is *not* given; in referring to any other Book, the number of the Book *is* given.

(P. IV.) OP and OQ form one and the same straight line; *which was to be proved.*

2.—*Given two lines, BE and AD: join B with D and A with E; and show that BD + AE is greater than BE + AD.*

In the triangle BCE, BC + CE > BE (P. VII.) and in the triangle DCA, CD + CA > AD; hence, BC + CD + CE + CA > BE + AD, but the first member of this inequality is equal to BD + AE, and therefore, BD + AE is greater than BE + AD: *which was to be proved.*



3.—*One of the two interior angles on the same side, formed by a straight line meeting two parallels, is one-half of a right angle; what is the other angle equal to?*

Since the sum of the two interior angles on the same side is equal to two right angles, (P. XX.), if one of them is one-half of a right angle, the other must be equal 1½ right angles.

4.—*The sum of two angles of a triangle is ⅘ of a right angle; what is the other angle equal to?*

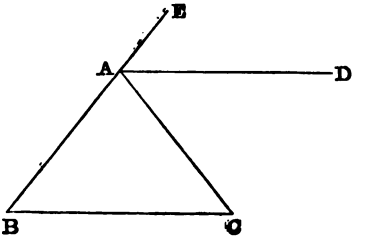
Since the sum of the three angles is equal to two right angles (P. XXV), the other angle must be equal to ⅘ of a right angle.

5.—*One of the acute angles of a right angled triangle is ⅗ of a right angle; what is the other?*

Since the sum of the acute angles of a right angled triangle is equal to a right angle (P. XXV, C. 4) the other acute angle must be ⅗ of a right angle.

6.—*Show that the line which bisects the exterior vertical angle of an isosceles triangle is parallel to the base of the triangle.*

Let ABC be an isosceles triangle, AB being equal to AC, CAE the exterior vertical angle, and AD the line which bisects that angle: then is AD parallel to BC.



The exterior angle CAE is equal to the sum of the interior and opposite angles, ABC and ACB (P. XXV, C. 6), and since ABC and ACB are equal (P. XI.), CAE = 2 ACB; and hence CAD, half of CAE, equals ACB: therefore, the two lines AD and BC are parallel (P. XIX, C. 1); *which was to be proved.*

7.—*The sum of the interior angles of a polygon is 12 right angles; what is the polygon?*

Two right angles taken as many times, less two, as this polygon has sides must be equal to 12 right angles (P. XXVI.); hence the number of sides *less two* is 6, and, therefore, the number of sides is 8, and the polygon is an octagon.

8.—*What is the sum of the interior angles of a heptagon equal to?*

The sum of the angles is equal to 2 right angles taken 7 — 2, or 5, times (P. XXVI.), *i. e.* 10 right angles.

9.—*The sum of five angles of a given equiangular polygon is 8 right angles; what is the polygon?*

Represent the number of sides and, consequently, the number of angles by n; then, since each angle is $\frac{4}{5}$ of a right angle, the sum of all the angles will be $\frac{4}{5}$ multiplied by the number of angles, or $\frac{4}{5}n$. But the sum of all the angles is equal to two right angles taken as many times, less two, as the polygon has sides (P. XXVI), or $2(n-2)$; hence,

$$2(n-2) = \tfrac{4}{5}n,$$

or

$$2n - 4 = \tfrac{4}{5}n.$$

Clearing of fractions, $10n - 20 = 8n$;

transposing and reducing, $2n = 20$;

hence, $n = 10$.

The polygon has 10 sides and is a decagon.

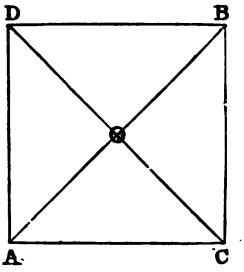
10.—*What part of a right angle is an angle of an equiangular decagon?*

The sum of all the angles of the decagon is 16 right angles (P. XXVI); hence, each angle is $\frac{16}{10}$ or $\frac{8}{5}$ of a right angle (P. XXVI, C. 4).

11.—*How many sides has a polygon in which the sum of the interior angles is equal to the sum of the exterior angles?*

Since the sum of the exterior angles is equal to 4 right angles (P. XXVII), the sum of the interior angles must be equal to 4 right angles; and, hence, the polygon is a quadrilateral (P. XXVI, C. 1).

12.—*Construct a square, having given one of its diagonals.*



Let AB be the given diagonal. Through the middle point, O, of AB draw CD perpendicular to AB, and make it equal to AB and bisected at O; join the extremities of these lines, and the resulting figure, ACBD, will be the square required; for

In the right-angled triangles AOD and DOB, AO = OB by construction, and OD is common, hence they are equal in all their parts (P. V), and AD = DB; in like manner it may be shown that DB = BC, and BC = CA; hence the figure is a parallelogram (P. XXIX) with all its sides equal. Again, in the triangles AOD and DOB, the angle ODA opposite the side OA is equal to the angle ODB opposite the equal side OB (P. X, S.); and since the angle OAD equals ODA (P. XI), OAD equals ODB. But the sum of the angles ODA and OAD is a right angle (P. XXV, C. 4), and, therefore, the sum of ODA and ODB, or ADB, is a right angle. In like manner, DBC and the remaining angles of the figure may be shown to be right angles. Therefore, ACDB is a square (D. 28, 1st) constructed on the given diagonal.

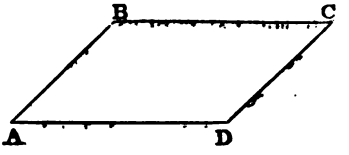
13.—*An angle is ¼ of a right angle; what is its complement and what its supplement?*

Since the *complement* of an angle is the difference between that angle and a *right angle*, and the *supplement* is the difference between that angle and *two right angles*, the complement

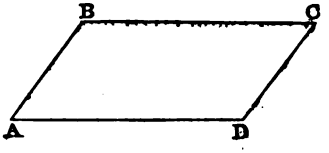
is $\frac{1}{4}$ of a right angle, and the supplement is $\frac{3}{4}$ of two right angles, or $\frac{3}{2}$ of one right angle.

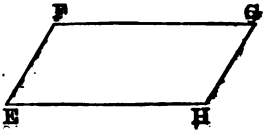
14.—*Show that any two adjacent angles of a parallelogram are supplements of each other.*

In the parallelogram ABCD, the sum of two adjacent angles, as BAD and ADC, is equal to two right angles (P. XX), and hence each is a supplement of the other.



15.—*Show that if two parallelograms have one angle in each equal, their remaining angles will be equal each to each.*

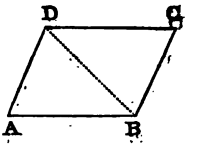




Let the two parallelograms have the angles A and E equal; then will the remaining angles be equal. The angles D and H will be equal, for they are supplements of the equal angles A and E; for a like reason B is equal to F, and C to G; *which was to be proved.*

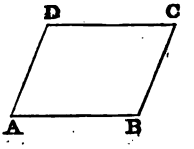
16.—*Show that if two sides of a quadrilateral are parallel and two opposite angles equal, the figure is a parallelogram.*

Let ABCD be a quadrilateral in which AD is parallel to BC, and the angle A is equal to the angle C; then is ABCD a par-



allelogram. Draw BD; then, since AD and BC are parallel and BD cuts them, the angle ADB equals the angle DBC (P. XX, C. 2); in the triangles ADB and DBC, the angle A equals the angle C by hypothesis, the angle ADB equals the angle DBC as just shown, therefore the angle ABD equals the angle BDC (P. XXV, C. 2); hence the sides AB and DC are parallel (P. XIX, C. 1), and hence the figure is a parallelogram (D. 28); *which was to be proved.*

17.—*Show that if the opposite angles of a quadrilateral are equal, each to each, the figure is a parallelogram.*



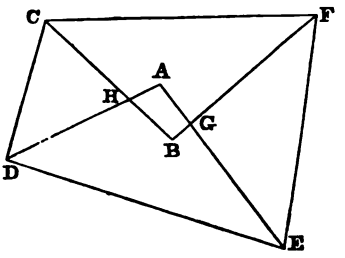
Let the angle A = angle C, and angle B = angle D; then is ABCD a parallelogram. Since A = C and B = D, the sum of A and B is equal to the sum of C and D, and since the sum of all the interior angles of the quadrilateral is equal to 4 right angles (P. XXVI, C. 1), the sum of A and B is two right angles; hence AD and BC are parallel (P. XIX). In like manner, AD and BC may be proved parallel: hence ABCD is a parallelogram (D. 28); *which was to be proved.*

18.—*Show that the lines which bisect the angles of any quadrilateral form, by their intersection, another quadrilateral, the opposite angles of which are supplements of each other.*

Let CDEF be a quadrilateral, and DA, EA, FB, and CB, the lines which bisect the angles.

The sum of the 4 angles of the quadrilateral is equal to 4

right angles (P. XXVI, C. 1); or, denoting the angles by the letters C, D, E, F, and 1 right angle by R, we have



$$C + D + E + F = 4R, \quad (1)$$

or by transposition,

$$C + F = 4R - (D + E);$$

and dividing by 2,

$$\tfrac{1}{2}(C + F) = 2R - \tfrac{1}{2}(D + E), \quad (2)$$

But ADE and AED are, respectively, equal to $\frac{1}{2}D$ and $\frac{1}{2}E$, and hence we have for the angle DAE (P. XXV, C. 1), denoted by A,

$$A = 2R - \tfrac{1}{2}(D + E); \quad . \quad . \quad . \quad . \quad . \quad (3)$$

hence we have, (A. 1), from (2) and (3),

$$A = \tfrac{1}{2}(C + F). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the same way, we have

$$B = \tfrac{1}{2}(D + E). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, from (4) and (5), by addition we have

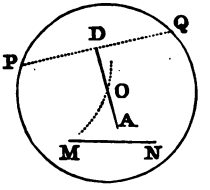
$$A + B = \tfrac{1}{2}(C + F + D + E).$$

Hence, (A. 1), $A + B = 2R$;

and, therefore, A and B are supplements of each other (p. 49**, Note 2); *which was to be proved.*

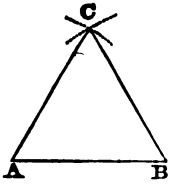
BOOK III.

1.—*Draw a circumference of a given radius through two given points.*



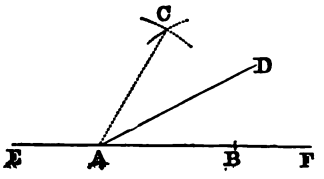
Let P and Q be the two given points, and let the given radius be equal to MN. Join P and Q by the line PQ, and bisect PQ by the perpendicular DA, the centre of the required circle will be somewhere upon the line DA (P. VI, C.). With one of the given points, as P, as a centre, and a radius equal to MN, describe an arc cutting DA in O; O is equally distant from P and Q (B. I, P. XVI), and is at a distance MN from each; with O as a centre, and a radius equal to MN, describe a circle, and it will be the circle required.

2.—*Construct an equilateral triangle, having given one of its sides.*



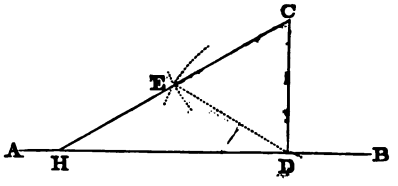
Let the given side be AB: from A as a centre, and with AB as a radius, describe an arc; then from B as a centre, and with the same radius, describe a second arc cutting the first at C; join C with A and B by the lines CA and CB; ABC is the triangle required.

3.—*At a point on a given straight line, construct an angle of 30°.*



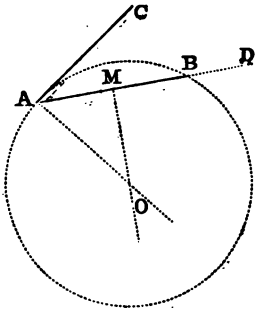
Let EF be the given line, and A a point of it; with A as a centre and any convenient radius, as AB, describe an arc; then with B as a centre and the same radius, describe a second arc intersecting the first at C; draw AC; BAC is equal to an angle of an equilateral triangle (see preceding exercise), and is therefore equal to $\frac{2}{3}$ of a right angle (B. I, P. XXV, C. 5), that is $\frac{2}{3}$ of 90°, or 60°; bisect the angle BAC (Prob. V) by the line AD; BAD, being half of BAC, is equal to 30° and is the angle required.

4.—*Through a given point without a given line, draw a line forming with the given line an angle of 30°.*



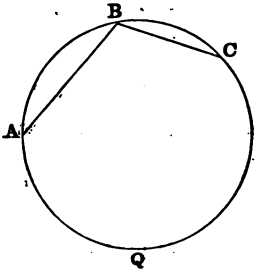
Let AB be the given line and C the given point. From C draw CD perpendicular to AB (Prob. III); from C and D as centres and a radius equal to CD, describe two arcs intersecting in E; draw DE and CE; then DEC is an equilateral triangle (E. 2, Key), and each of its angles is equal to $\frac{2}{3}$ of a right angle, or 60° (B. I, P. XXV, C. 5); prolong CE to H, then is CH the line required. For in the right-angled triangle CHD, the sum of the two acute angles DHC and DCH, is equal to a right angle (B. I, P. XXV, C. 4), or 90°; and DCH being equal to 60°, DHC must be equal to 90° — 60°, or 30°.

5.—A line 8 feet long is met at one extremity by a second line, making with it an angle of 30°; find the centre of the circle of which the first line is a chord and the second a tangent.



Draw an indefinite line AD, and on it, on any convenient scale,* lay off AB equal to 8 feet; from A draw AC, making with AD an angle of 30° (E. 3); this will be the second line. From M, the middle point of AB, draw a perpendicular to AB; it will pass through the required centre (P. VI, C.); at A draw a perpendicular to AC, it will also pass through the required centre (P. IX); and since the centre is upon both the lines, it must be at their point of intersection, O; hence O is the centre required.

6.—*How many degrees in an angle inscribed in an arc of 135°?*



Let ABC be inscribed in an arc of 135°. Since the arc ABC contains 135°, and the whole circumference contains 360°, the arc AQC must contain 225°; as ABC is measured by half the arc AQC (P. XVIII), it contains $\frac{1}{2}$ of 225°, or 112$\frac{1}{2}$°.

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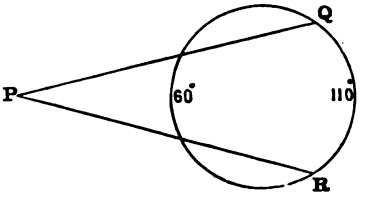
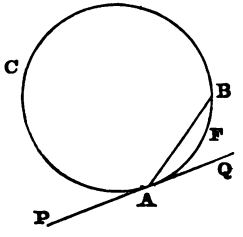
* In making the above and similar constructions, some convenient length is taken as a unit, or 1. In the above, $\frac{1}{8}$ of an inch represents a foot, and, consequently, 1 inch would represent 8 feet, and the scale is said to be 8 feet to the inch.

7.—How many degrees in the angle formed by two secants meeting without the circle and including arcs of 60° and 110°?



The angle QPR is measured by half the difference of the intercepted arcs (P. XX); hence, angle QPR $= \frac{1}{2}(110° - 60°) = 25°$.

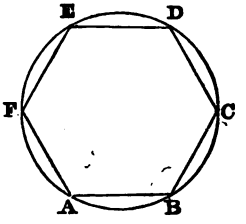
8.—At one extremity of a chord, which divides the circumference into two arcs of 290° and 70° respectively, a tangent is drawn; how many degrees in each of the angles formed by the tangent and the chord?



The angle QAB is measured by half the arc AFB (P. XXI); that is, half of 70° or 35°.

The angle PAB is measured by half the arc ACB; *i. e.*, half of 290°, or 145°.

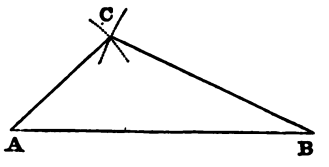
9.—Show that the sum of the alternate angles of an inscribed hexagon is equal to four right angles.



The angle ABC is measured by half the arc AFEDC (P. XVIII); CDE is measured by half the arc EFABC; EFA is measured by half the arc ABCDE; hence, by addition, ABC + CDE + EFA is measured by $\frac{1}{2}$(AFEDC + EFABC + ABCDE); but the quantity in the parenthesis is equal to twice the circumference of the circle ACF,

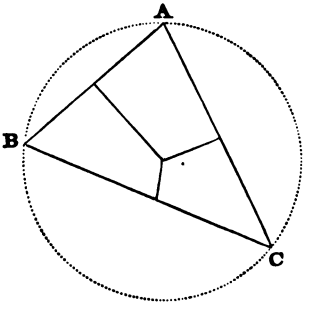
and half of it is equal to the circumference; hence the sum of the alternate angles ABC, CDE, EFA, being measured by the circumference of a circle, is equal to 4 right angles (P. XVII, S.); *which was to be proved.*

10.—*The sides of a triangle are 3, 5, and 7 feet; construct the triangle.*



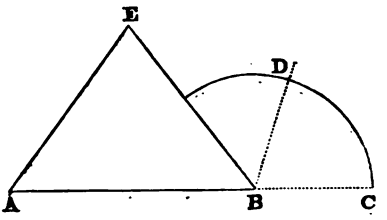
Draw AB on any scale (See E. 5, Key), and make it equal to 7 feet. With A as a centre and a radius equal to 3 feet, describe an arc; then with B as a centre and a radius equal to 5 feet, describe a second arc; from the point of intersection of the two arcs, C, to A and B, draw the lines CA and CB; ABC will be the triangle required.

11.—*Show that the three perpendiculars erected at the middle points of the three sides of a triangle meet in a common point.*



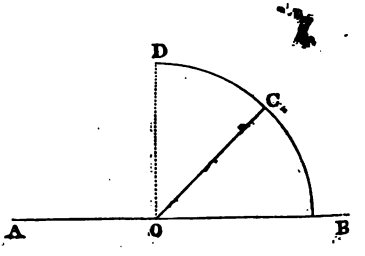
Let ABC be the triangle. Through its three vertices pass the circumference of a circle (P. VII), the sides of the triangle will then be chords of this circle; at the middle point of each of the sides, erect a perpendicular; each perpendicular will pass through the centre of the circle (P. VI, C.), and, hence, the three perpendiculars will meet in a common point; *which was to be proved.*

12.—*Construct an isosceles triangle with a given base and a given vertical angle.*



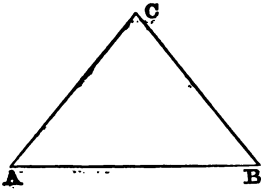
Let AB be the given base. Prolong AB to C, and at B construct the angle CBD, equal to the given vertical angle (Prob. IV); bisect the angle DBA (Prob. V) by the line BE; at the point A construct the angle BAE equal to the angle ABE, then will AEB be the angle required. For, since the angle A is equal to the angle ABE, the side EA is equal to the side EB (B. I, P. XII), and the triangle is isosceles; the angle EBC is equal to the sum of the angles A and E (B. I, P. XXV, C. 6); the angle EBA is equal to the angle EBD by construction, and, therefore, the angle A is equal to EBD; hence the angle E is equal to the angle DBC, which is by construction equal to the given vertical angle.

13.—*At a point on a given straight line, construct an angle of 45°.*



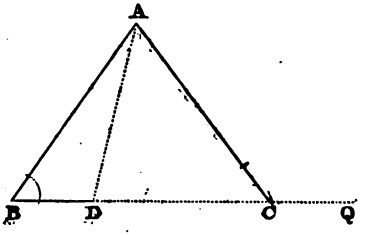
Let AB be the given line and O a point on the line. At O, draw the perpendicular OD (Prob. II); bisect the angle BOD (Prob. V) by the line OC, and BOC being half a right angle, is an angle of 45°.

14.—*Construct an isosceles triangle so that the base shall be a given line and the vertical angle a right angle.*



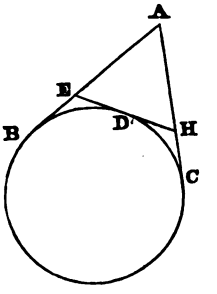
Let AB be the given base. At A and B draw lines, each making with the line AB an angle of 45°, and prolong them till they meet in C, then will ACB be the required triangle; for since the angles A and B are equal, the triangle is isosceles (B. 1, P. XII); and since A and B are each 45°, their sum is 90°, or a right angle; and as the sum of the three angles is equal to two right angles, the remaining angle C is a right angle.

15.—*Construct a triangle, having given one angle, one of its including sides, and the difference of the two other sides.*



Draw the indefinite line BQ; on it lay off BD equal to the given difference of the two sides; at B, on the line BQ, construct the angle DBA equal to the given angle (Prob. IV), and draw BA, making it equal to the given including side; draw AD; at A, on the line DA, construct the angle DAC equal to the QDA, and draw AC; then AC is equal to DC (B. I, P. XII), and BC — AC equals BC — CD, or BD, the given difference of the two sides; hence BAC is the triangle required.

16.—*From a given point, A, without a circle, draw the tangents, AB and AC, and at any point, D, in the included arc, draw a third tangent and produce it to meet the two others; show that the three tangents form a triangle whose perimeter is constant.*



For, $AH = AC - HC,$

$$AE = AB - EB,$$

$$HE = HD + ED,$$

and by addition,

$$AH + AE + HE = AC - HC + AB - EB + HD + ED;$$

or, by changing the order of terms in the second member of the equation,

$$AH + AE + HE = AC + HD - HC + AB + ED - EB.$$

But from (Prob. XIV, C.), $HD = HC$ and $ED = EB$; hence, by reduction,

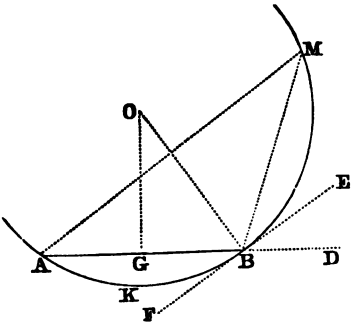
$$AH + AE + HE = AC + AB;$$

that is, the perimeter of the triangle is equal to the sum of the two tangents drawn from A, a constant quantity; *which was to be proved.*

17.—*On a straight line 5 feet long, construct a circular segment that shall contain an angle of 30°.*

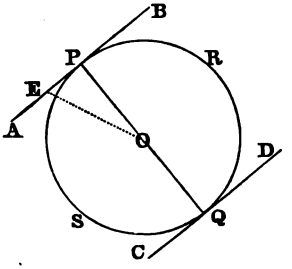
Draw AB and make it equal to five feet in length, according to the scale adopted. Prolong AB towards D; at B construct the angle DBE equal to 30° (E. 3); draw BO perpendicular to BE (Prob. II), and at the middle

point G, of AB, draw GO perpendicular to AB; from their point of intersection, O, as a centre, with a radius OB, describe the arc AMB, then will the segment AMB be the segment required. For the angle ABF, equal to EBD (B. I, P. II), is measured by half the arc AKB (P. XXI), and the inscribed angle AMB is measured by half the same arc (P. XVIII); hence the angle AMB is equal to the angle EBD, and, consequently, to 30°.



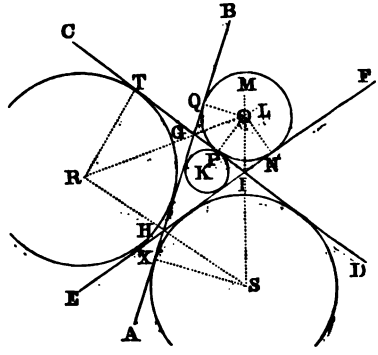
18.—*Show that parallel tangents to a circle include semicircumferences between their points of contact.*

Let AB and CD be parallel tangents to the circle whose centre is O; then will the arcs PRQ and PSQ, contained between their points of contacts, P and Q, be semicircumferences. For, from the centre, O, draw OQ to the point of contact of CD; it will be perpendicular to CD (P. IX); prolong QO till it meets the tangent AB, then will QO prolonged be perpendicular to AB (B. I, P. XX, C. 1); since QO prolonged is perpendicular to AB, it must meet AB at the point of contact P, for if it does not, let it meet AB in E; draw OP to the point of contact and it will be perpendicular to AB



(P. IX), and we shall have two perpendiculars, OP and OE, drawn from the same point, O, to the same straight line, AB, which is impossible (B. 1, P. XIV); hence QO prolonged must meet AB at the point of contact P; therefore, the line PQ joining the points of contact of the parallel tangents is a diameter of the circle, and the arcs, PRQ and PSQ, included between the points of contact, are semicircumferences (P. I); *which was to be proved.*

19.—*Show that four circles can be drawn tangent to three intersecting straight lines.*

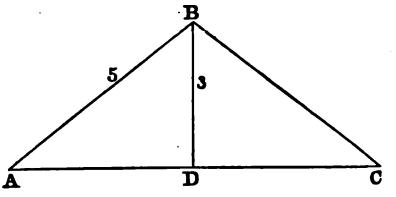


Let AB, CD, and EF, be three lines which intersect at the points G, H, and I. In the triangle GHI, inscribe the circle whose centre is K (Prob. XV). Bisect the angles BGI and GIF (Prob. V) by the lines GL and IM, and from their point of intersection, O, draw OQ, OP, and and ON perpendicular, respectively, to GB, GI, and IF; in the right-angled triangles OGQ and OGP, the angle OQG equals the angle OPG, since each is a right angle; OGQ equals OGP by construction; therefore the remaining angles, QOG and GOP, are equal (B. I, P. XXV, C. 2); the side OG is common, and therefore the triangles are equal in all their parts (B. I, P. VI); hence OQ is equal to OP; in like manner, OP may be shown equal to ON; then with O as a centre and OQ as a radius, describe a circle; it

will be tangent to each of the three given lines (P. IX). In like manner, bisect the angles CGH and GHE by the lines GR and HR, and with their point of intersection, R, as a centre, and a radius equal to RT, the perpendicular distance to any one of the given lines, describe a circle; it will be tangent to the three given lines; bisect, also, the angles AHI and HID by the lines HS and IS, and with their point of intersection, S, as a centre, and SX as a radius, describe the fourth circle tangent to the three given intersecting lines.

BOOK IV.

1.—*The altitude of an isosceles triangle is 3 feet, each of the equal sides is 5 feet; find the area.*



Let ABC be the triangle, in which AB is 5 feet and BD is 3 feet. AD is one-half the base (B. I, P. XI, C. 2). $\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2$ (P. XI), hence $\overline{AD}^2 = 25 - 9 = 16$, and AD = 4; the area of ABC is equal to AD multiplied by DB (P. VI), and hence is equal to $4 \times 3 = 12$ square feet; which is the area required.

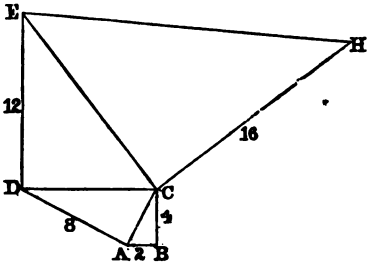
2.—*The parallel sides of a trapezoid are 8 and 10 feet, and the altitude is 6 feet; what is the area?*

From P. VII, the area is equal to the altitude multiplied by half the sum of its parallel sides; *i. e.*, to $6 \times \frac{8 + 10}{2} = 6 \times 9 = 54$ square feet.

3.—*The sides of a triangle are 60, 80, and 100 feet; the diameter of the inscribed circle is 40 feet; find the area.*

The diameter of the inscribed circle being 40 feet, the radius is 20 feet; hence (P. VI, C. 2) the area is equal to $\frac{60 + 80 + 100}{2} \times 20 = 120 \times 20 = 2400$ square feet.

4.—Construct a square equal to the sum of the squares whose sides are, respectively, 16, 12, 8, 4, and 2 units in length.

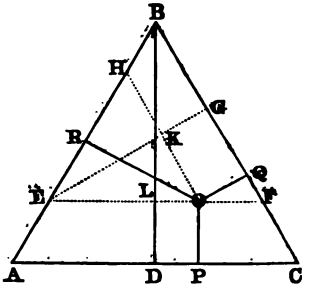


Take any convenient scale, and lay off AB equal to 2 units of it; at B erect BC perpendicular to AB, and make it equal to 4 units of the scale; AC will be the side of a square equal to the sum of the squares whose sides are 4 and 2 (P. XI). At A erect AD perpendicular to AC and make it equal to 8 units, and draw DC; DC will be the side of a square equal to the sum of the squares whose sides are AC and 8, *i. e.*, 2, 4, and 8. At D erect DE perpendicular to DC and make it equal to 12 units of the scale, and draw EC; EC will be the side of a square equal to the sum of the squares whose sides are DC and 12, *i. e.*, 2, 4, 8, and 12. At C erect CH perpendicular to EC, and make it equal to 16 units; EH will be the side of a square equal to the sum of the squares whose sides are EC and CH; *i. e.*, 2, 4, 8, 12, and 16, and hence is the side of the required square; on EH, as a side, construct a square by drawing at E and H straight lines perpendicular to EH and equal to it, and connecting the extremities of the perpendiculars, and it will be the square required.

Or, the required construction may be made without making the several separate constructions, by adding together the squares of the several given sides, and taking the square root of the result as the side of the required square. The sum of the

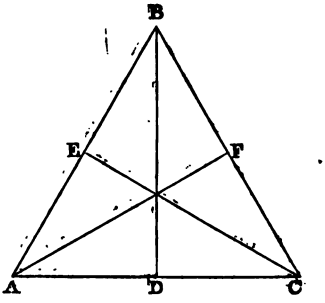
squares of the given sides is 484, the square root of which is 22. Then draw a line, as EH, equal to 22 units of the scale taken and construct on it a square, as before.

5.—*Show that the sum of the three perpendiculars, drawn from any point within an equilateral triangle to the three sides, is equal to the altitude of the triangle.*



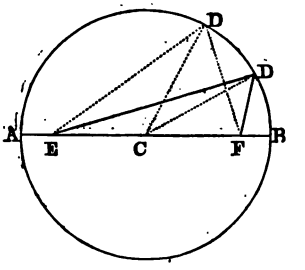
Let ABC be an equilateral triangle, and BD its altitude. From O, a point within the triangle, draw OP, OQ, and OR, perpendicular to the sides AC, CB, and BA, respectively; then will OP + OQ + OR = BD. Through O draw EF parallel to AC (B. III, Prob. VI); from O draw OH parallel to BC; from E draw EG parallel to OQ, it will be perpendicular to BC and its parallel OH (B. I, P. XX, C. 1). In the triangles ABC and EOH, we have the angle EOH equal the angle ACB (B. I, P. XXIV), the angle HEO equal the angle BAC, and EHO equal ABC (B. I, P. XX, C. 3); hence the two triangles are mutually equiangular and, therefore, similar (P. XVIII); consequently, since the triangle ABC is equilateral, EOH is also equilateral, and its altitude EK is equal to its altitude OR (see Note at the end); KG is equal to OQ (B. I, P. XXVIII, C. 2); therefore, the sum of EK and KG, or EG, is equal to the sum of the two perpendiculars OR and OQ. The triangles BEF and BAC have the angle B common, the angles BEF and BAC equal, and BFE equal to BCA (B. I, P. XX,

C. 3); therefore they are mutually equiangular and hence similar (P. XVIII); consequently, BEF is an equilateral triangle, and its altitude EG is equal to its altitude BL; hence, BL is equal to the sum of the perpendiculars OR and OQ; LD is equal to OP (B. I, P. XXVIII, C. 2); therefore, BD, which is equal to BL + LD, is equal to the sum of the three perpendiculars OR, OQ, and OP; *which was to be proved.*



NOTE.—The three altitudes of an equilateral triangle are equal to each other. For, in the right-angled triangles BDC and AFC, BC = AC, by hypothesis; FC = DC, being halves of equal sides; hence, the triangles are equal in all their parts (B. I, P. XVII); hence, AF = BD; in like manner, CE = BD.

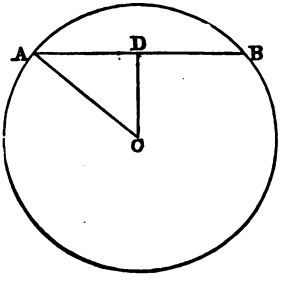
6.—*Show that the sum of the squares of two lines, drawn from any point in the circumference of a circle to two points on the diameter of the circle equidistant from the centre, will be always the same.*



Let the points E and F be equidistant from the centre, C, of the circle ADB; draw DE and DF; then will $\overline{DE}^2 + \overline{DF}^2$ be a constant quantity; *i. e.*, will always be the same, no matter at what point of the circumference D is taken. Draw CD; since the point E is fixed, CE remains constant, and CD, being the radius, remains constant

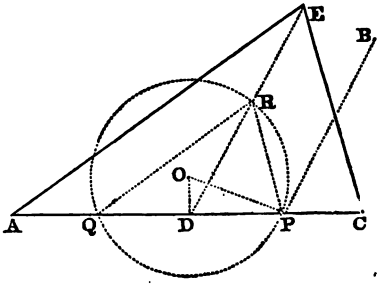
no matter at what point of the circumference D is taken; hence, $\overline{DE}^2 + \overline{DF}^2$ equal to $2\overline{CE}^2 + 2\overline{CD}^2$ (P. XIV), is the same for every point of the circumference; *which was to be proved.*

7.—*The distance of a chord, 8 feet long, from the centre of a circle, is 3 feet; what is the diameter of the circle?*



Let the chord AB be 8 feet long, and the distance CD, 3 feet; CD bisects the chord AB (B. III, P. VI), hence AD is equal to 4 feet. In the right-angled triangle CDA, $\overline{CA}^2 = \overline{CD}^2 + \overline{DA}^2$ (P. XI); hence $\overline{CA}^2 = 9 + 16 = 25$, and CA, the radius of the circle, is equal to 5 feet; and hence the diameter, which is twice the radius, is 10 feet.

8.—*Construct a triangle, having given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base.*



Draw AC; at some point of it, as D, draw DE, making with CA an angle equal to the given angle, which the bisecting line makes with the base (B. III, Prob. IV); lay off DP equal to DQ, any convenient length; on QP construct a segment QRP, which shall contain an angle equal to the given vertical

angle (B. III, Prob. XVI); from the point R, where the arc of the segment cuts the line DE, draw RP and RQ; from E, draw EC and EA, respectively parallel to RP and RQ; then is AEC the triangle required. For, the angle AEC is equal to the angle QRP (B. 1, P. XXIV), and hence to the given vertical angle; the triangles ADE and QDR, since QR is parallel to AE, are similar (P. XVIII), and

$$AD : QD :: DE : DR; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for a like reason the triangles CDE and PDR are similar, and

$$CD : PD :: DE : DR. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) we have, (B. II, P. IV),

$$AD : QD :: CD : PD.$$

But $$QD = PD;$$

hence $$AD = CD;$$

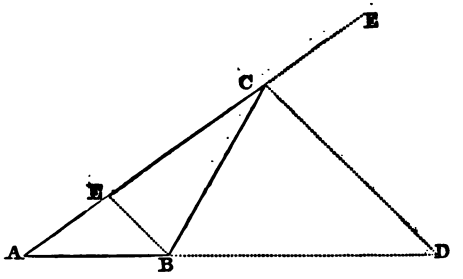
hence D is the middle point of the base; the triangle AEC, therefore, fulfils all the conditions and is the triangle required.

9.—*Show that if a line bisecting the exterior vertical angle of a triangle is not parallel to the base, the distances of the point in which it meets the base produced, from the extremities of the base, are proportional to the other two sides of the triangle.*

Let ABC be the triangle and CD the line which bisects the exterior vertical angle BCE; then,

$$AD : BD :: AC : BC.$$

From B draw BE parallel to DC; then, the angle CBE equals



the angle BCD (B. I, P. XX, C. 2); but BCD equals DCE by construction; hence angle CBE equals angle DCE; but DCE equals CEB (B. I, P. XX, C. 3); hence the angles CEB and CBE are equal (A. 1), and, therefore, the line CB equals CE (B. I, P. XII). In the triangle ACD, since BE is parallel to the base,

$$AD : AB :: AC : AE \text{ (P. XV, C. 1)};$$

hence, by division (B. II, P. VI),

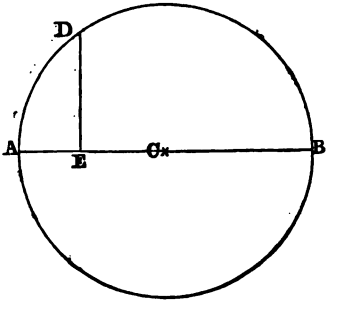
$$AD : AD - AB, \text{ or } DB :: AC : AC - AE, \text{ or } EC;$$

but, since EC equals BC, we have

$$AD : BD :: AC : BC;$$

which was to be proved.

10.—*The segments made by a perpendicular, drawn from a point on the circumference of a circle to a diameter, are 16 feet and 4 feet; find the length of the perpendicular.*



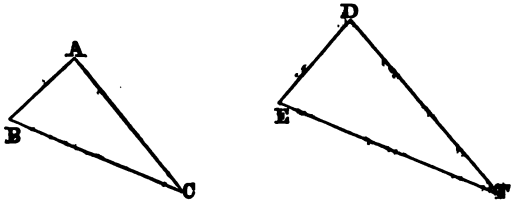
In a circle whose centre is C, let the perpendicular DE be drawn, making AE equal to 4 feet and EB equal to 16 feet. From Prop. XXV, C. 2, we have

$$AE : ED :: ED : EB;$$

hence $$\overline{ED}^2 = AE \times EB = 4 \times 16 = 64,$$

and $$ED = 8 \text{ feet.}$$

11.—*Two similar triangles, ABC and DEF, have the homologous sides AC and DF equal, respectively, to 4 feet and 6 feet, and the area of DEF is 9 square feet; find the area of ABC.*



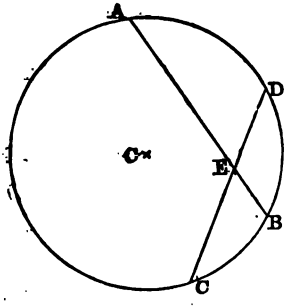
From P. XXV,

$$\text{Area DEF} : \text{Area ABC} :: \overline{DF}^2 : \overline{AC}^2;$$

hence, $$\text{Area ABC} = \frac{\text{Area DEF} \times \overline{AC}^2}{\overline{DF}^2} = \frac{9 \times 16}{36} = 4;$$

therefore, the area of ABC is 4 square feet.

12.—*Two chords of a circle intersect; the segments of one are, respectively, 6 feet and 8 feet, and one segment of the other is 12 feet; find the remaining segment.*



In the circle whose centre is C, let the chords AB and CD intersect at E, making DE equal to 6 feet;

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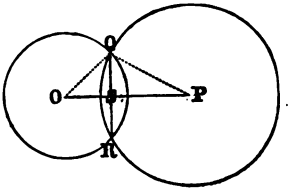
EC, 8 feet; and AE, 12 feet; find EB. From P. XXVIII, we have

$$AE : DE :: EC : BE;$$

hence $$BE = \frac{DE \times EC}{AE} = \frac{6 \times 8}{12} = 4 \text{ feet.}$$

13.—*Two circles, whose radii are 6 feet and 10 feet, intersect, and the line joining their points of intersection is 8 feet; find the distance between their centres.*



Let the circles whose centres are O and P, and whose radii are respectively 6 feet and 10 feet, intersect, and let the line QR, joining their points of intersection, be 8 feet. Join the centres by the line OP, intersecting QR in S; draw OQ and PQ. QS is one-half of QR (B. III, P. XI), and is therefore, equal to 4 feet. In the right-angled triangle QSP, $\overline{SP}^2 = \overline{QP}^2 - \overline{QS}^2$ (P. XI) $= 100 - 16 = 84$, and $SP = \sqrt{84} = 9.16$. In like manner, $\overline{OS}^2 = \overline{OQ}^2 - \overline{QS}^2 = 36 - 16 = 20$, and $OS = \sqrt{20} = 4.47$; therefore OP, the distance between the centres, equals $9.16 + 4.47 = 13.63$ feet.

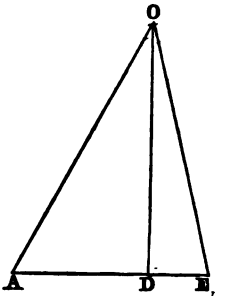
14.—*Find the area of a triangle whose sides are respectively 31, 28, and 20 feet.*

In the triangle AOE, let AO = 31 ft, OE = 28 ft., and AE = 20 ft. Draw OD perpendicular to AE. Then (P. XI),

$$\overline{OD}^2 = \overline{AO}^2 - \overline{AD}^2,$$

and $$\overline{OD}^2 = \overline{OE}^2 - \overline{ED}^2;$$

hence $\overline{AO}^2 - \overline{AD}^2 = \overline{OE}^2 - \overline{ED}^2$;



by transposition,

$$\overline{AO}^2 - \overline{OE}^2 = \overline{AD}^2 - \overline{ED}^2;$$

hence (P. X),

$$(AO + OE)(AO - OE) = (AD + ED)(AD - ED);$$

and substituting the numerical values, we have

$$59 \times 3 = 20\,(AD - ED);$$

hence $AD - ED = \frac{177}{20} = 8.85$;

therefore, since $AD + ED = 20$,

and $AD - ED = 8.85$,

we have by addition, $2AD = 28.85$,

and $AD = 14.425$;

and by subtraction $2ED = 11.15$,

and $ED = 5.575$.

In the right-angled triangle ODE, we have

$$\overline{OD}^2 = \overline{OE}^2 - \overline{ED}^2,$$

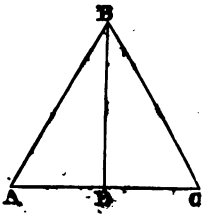
or substituting numerical values,

$$\overline{OD}^2 = (28)^2 - (5.575)^2 = 752.92,$$

and $$OD = \sqrt{752.92} = 27.44.$$

The area of the triangle AOE is equal to $\frac{1}{2}AE \times OD$ (P. VI); *i. e.*, to $\frac{1}{2} \times 20 \times 27.44$, which is 274.4 square feet, the area required.

15.—*Show that the area of an equilateral triangle is equal to one-fourth the square of one side multiplied by* $\sqrt{3}$; *or to the square of one side multiplied by .433.*



Let ABC be an equilateral triangle, and BD its altitude. D is the middle point of the base (B. 1, P. XI, C. 2), and hence

$$AD = \tfrac{1}{2}AC, \text{ or } \tfrac{1}{2}AB.$$

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 \text{ (P. XI, C. 1);}$$

hence $$\overline{BD}^2 = \overline{AB}^2 - \tfrac{1}{4}\overline{AB}^2 = \tfrac{3}{4}\overline{AB}^2,$$

and $$BD = \tfrac{1}{2}AB\sqrt{3}.$$

But the area of the triangle is equal to

$$\tfrac{1}{2}AC \times BD \text{ (P. VI), or } \tfrac{1}{2}AB \times BD,$$

and hence the

$$\text{Area ABC} = \tfrac{1}{2}AB \times \tfrac{1}{2}AB\sqrt{3} = \tfrac{1}{4}\overline{AB}^2\sqrt{3};$$

or since $$\sqrt{3} = 1.732,$$

$$\text{Area ABC} = \tfrac{1}{4}\overline{AB}^2 \times 1.732 = \overline{AB}^2 \times .433;$$

which was to be proved.

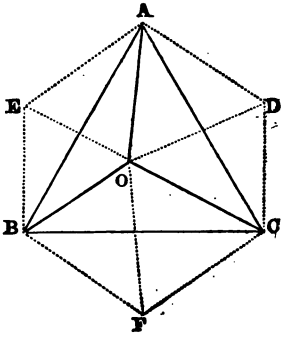
Cor.—Since $$\text{Area ABC} = \overline{AB}^2 \times .433$$

we have $$\overline{AB}^2 = \frac{\text{Area ABC}}{.433},$$

and $$AB = \sqrt{\frac{\text{Area ABC}}{.433}}$$

that is, the side of an equilateral triangle is equal to the square root of the quotient arising from dividing the area of the triangle by .433.

16.—*From a point, O, in an equilateral triangle, ABC, the distances to the vertices were measured and found to be: OB = 20, OA = 28, OC = 31; find the area of the triangle and the length of each side.*



On AC construct a triangle having the side AD = OA = 28 and CD = OB = 20; on BC construct BFC, having CF = OC = 31, and BF = OA = 28; and on BA construct BEA, having BE = OB = 20, and AE = OC=31; draw OD, OE, and OF. The triangles AEB and BOC have the sides AE and OC, and also BE and OB, equal by construction, and AB equal to BC by hypothesis; hence they are equal in all their parts (B. I, P. X). In like manner, the triangles ADC and AOB are equal, having the sides AD and AO, CD and OB, equal by construction, and AC and AB equal by hypothesis; and the triangles BFC and AOC are equal, having BF and OA, CF and OC, equal by construction, and BC and AC equal by hypothesis; hence the area of the triangle ABC is one-half the area of the polygon AEBFCD. Since the triangles ADC and AOB are equal in all their parts, the angle CAD equals the angle OAB (B. I, P. X, S.); hence the sum of the angles CAD and CAO, or DAO, is equal to the sum of the angles BAO and CAO, or BAC; but BAC, being an angle of an equilateral triangle, is $\frac{2}{3}$ of a right angle (B. 1, P. XXV, C. 5), and, therefore, the angle DAO is $\frac{2}{3}$ of a right angle; in the triangle ADO, AO equals AD by construction,

and hence the angle ADO equals the angle AOD (B. I, P. XI), and since the sum of the two is $\frac{1}{3}$ of a right angle (OAD being $\frac{1}{3}$ of a right angle), each is $\frac{1}{3}$ of a right angle, and the triangle is equiangular and hence equilateral (B. I, P. XII, C.). In like manner, the triangles COF and BOE may be shown to be equilateral. The triangles AOE, BOF, and COD, have AE = OF = OC, OA = BF = OD, OE = OB = CD, and hence are equal in all their parts (B. I, P. X), and their sum is equal to 3AOE; the polygon AEBFCD is, therefore, equal to AOD + COF + BOE + 3AOE.

AOD being an equilateral triangle, each side of which is 28, is equal in area to $(28)^2 \times .433 = 339.472$ (see preceding exercise); in like manner, COF equals $(31)^2 \times 433 = 416.113$, and BOE equals $(20)^2 \times .433 = 173.2$; the area of AOE is 274.4 (E. 14, Key), and 3AOE = 823.2; hence the area of the polygon AEBFCD is equal to 339.472 + 416.113 + 173.2 + 823.2 = 1751.985. The triangle ABC is one-half the polygon AEBFCD, and is, therefore, equal to $\frac{1751.985}{2} = 875.9925$; *which was to be found* in the first place.

ABC being an equilateral triangle, one side of it is equal to

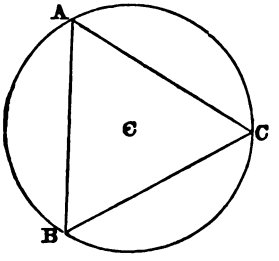
$$\sqrt{\frac{\text{Area ABC}}{.433}} \text{ (E. 15, C.); } i.\ e.\text{, to}$$

$$\sqrt{\frac{875.9925}{.433}} = \sqrt{2023.0773} = 44.978+;$$

which was to be found in the second place.

BOOK V.

1.—*The side of an equilateral triangle inscribed in a circle is 6 feet; find the radius of the circle.*

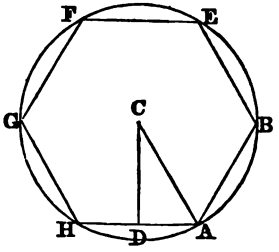


Let ABC be an equilateral triangle inscribed in a circle whose centre is C, and let s denote a side of the triangle and r the radius of the circle. Then from P. V, C. 2,

$$s : r :: \sqrt{3} : 1;$$

hence $r = \frac{s}{\sqrt{3}} = \frac{6}{1.732} = 3.46$ feet.

2.—*The radius of a circle is 10 feet; find the apothem of a regular inscribed hexagon.*



Let HE be a regular hexagon inscribed in the circle whose centre is C. The side AH is equal to the radius (P. IV); AD is one-half of AH (B. III, P VI), and hence is equal to 5 feet. In the right-angled triangle CDA,

$$\overline{CD}^2 = \overline{CA}^2 - \overline{AD}^2 \quad \text{(B. IV, P. XI, C. I);}$$

that is, $\overline{CD}^2 = 100 - 25 = 75.$

hence, $CD = \sqrt{75} = 8.66$ feet.

3.—*Find the side of a square inscribed in a circle whose radius is 5 feet.*

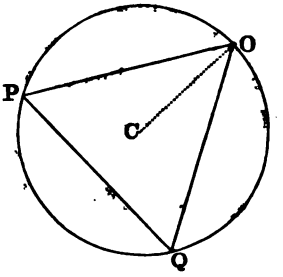
Let s denote the side of the square, and r the radius of the circle. Then from P. III, S.,

$$r : s :: 1 : \sqrt{2};$$

hence, $$s = r\sqrt{2} = 5 \times 1.414 = 7.07 \text{ feet.}$$

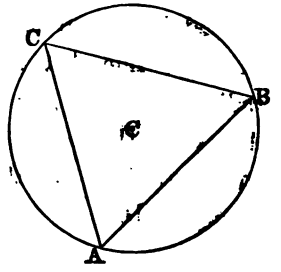
4.—*Draw a line whose length shall be* $\sqrt{3}$.

With C as a centre, and CO, equal to 1 (any convenient unit of length), as a radius, describe a circle, and in it inscribe an equilateral triangle (P. V, C. 1); any side of it, as OP, will be equal to $\sqrt{3}$ (P. V, C. 2).



5.—*The radius of a circle is 4 feet; find the area of an inscribed equilateral triangle.*

Let C be the centre of a circle whose radius is 4 feet, and ABC an inscribed equilateral triangle. Let s denote the side of the triangle and r the radius of the circle; then (P. V, C. 2),



$$s : r :: \sqrt{3} : 1$$

and hence $$s = r\sqrt{3} = 4\sqrt{3}.$$

But (B. IV, E. 15, Key),

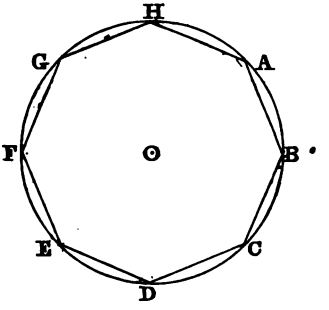
Area ABC $= s^2 \times .433 = 48 \times .433 = 20.784$ square feet;

which is the area required.

6.—*Show that the sums of the alternate angles of an octagon inscribed in a circle are equal to each other.*

Let ABCDE be an octagon inscribed in a circle whose centre is O; then will the sums of the alternate angles be equal to each other; *i. e.*,

$$A+G+E+C = H+F+D+B.$$



For the

$$\text{Angle A} = \tfrac{1}{2} \text{ (arc HFDB)}$$

(B. III, P. XVI).

But arc HFDB equals the circumference diminished by arc BAH, and, denoting the circumference by circ., we have

$$\text{Angle A} = \tfrac{1}{2} \text{ (circ. — arc BAH).}$$

In like manner,

$$\text{Angle G} = \tfrac{1}{2} \text{ (arc FDBH)} = \tfrac{1}{2} \text{ (circ. — arc HGF);}$$

$$\text{Angle E} = \tfrac{1}{2} \text{ (arc DBHF)} = \tfrac{1}{2} \text{ (circ. — arc FED);}$$

$$\text{Angle C} = \tfrac{1}{2} \text{ (arc BHFD)} = \tfrac{1}{2} \text{ (circ. — arc DCB).}$$

Hence, by addition,

$$A + G + E + C = \tfrac{1}{2} [4 \text{ circ.} - (BAH + HGF + FED + DCB)]$$
$$= \tfrac{1}{2} (4 \text{ circ.} - \text{circ.}) = \tfrac{3}{2} \text{ circ.};$$

and since each $\frac{1}{2}$ circ. measures 2 right angles (B. III, P. XVII, S.),

$$A + G + E + C = 6 \text{ right angles.}$$

But the sum of the interior angles of an octagon is equal to 12 right angles (B. I, P. XXVI), and, therefore, H + F + D + B is equal to 6 right angles, and hence equal to A + G + E + C; *which was to be proved.*

7.—The area of a regular hexagon, whose side is 20 feet, is 1039.23 square feet; find the apothem.

Denoting the area by A, the perimeter by p, and the apothem by h, we have

$$A = \frac{p \times h}{2} \text{ (P. VIII);}$$

hence, $$h = \frac{2A}{p} = \frac{2 \times 1039.23}{20 \times 6} = \frac{2078.46}{120} = 17.32;$$

which is the apothem required.

8.—One side of a regular decagon is 20 feet, and its apothem 15.4 feet; find the perimeter and the area of a similar decagon whose apothem is 8 feet.

The perimeter of a decagon is

$$20 \times 10 = 200 \text{ feet;}$$

its area is $\frac{200 \times 15.4}{2} = 1540$ square feet (P. VIII).

The perimeters of the two decagons are to each other as their apothems, and their areas as the squares of their apothems (P. IX); hence, letting p represent the perimeter of the second decagon, and A its area, we have

$$200 : p :: 15.4 : 8;$$

$$1540 : A :: 237.16 : 64;$$

hence, $$p = \frac{200 \times 8}{15.4} = 103.89+,$$

which is the perimeter required; and

$$A = \frac{1540 \times 64}{237.16} = 415.58+,$$

which is the area required.

9.—*The area of a regular hexagon inscribed in a circle is 9 square feet, and the area of a similar circumscribed hexagon is 12 square feet; find the areas of regular inscribed and circumscribed polygons of 12 sides.*

Denote the area of the inscribed polygon of 12 sides by p, and that of the similar circumscribed polygon by P; then (P. XI),

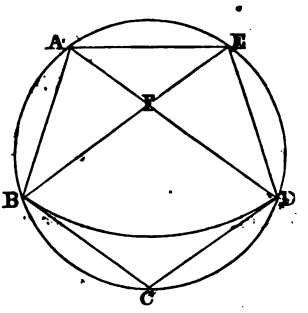
$$p = \sqrt{9 \times 12} = \sqrt{108} = 10.39 \text{ square feet};$$

$$P = \frac{2 \times 9 \times 12}{9 + 10.39} = 11.139 \text{ square feet};$$

which was to be done.

10.—*Given two diagonals of a regular pentagon that intersect; show that the greater segments will be equal to each other and to a side of the pentagon, and that the diagonals cut each other in extreme and mean ratio.*

Let ABCDE be a regular pentagon, and let the diagonals BE and AD intersect at F. Circumscribe a circle about the



pentagon; the arcs AB and ED are equal (B. III, P. V), as also are the arcs BC and CD; hence the arc ABC, which is equal to the sum of the arcs AB and BC, is equal to the arc CDE, the sum of the arcs CD and DE; the angle ADC, measured by one-half the arc ABC (B. III, P. XVIII), is equal to the angle EBC, measured by half the equal arc CDE. The angle BFD is measured by half the sum of the arcs BCD and AE (B. III, P. XIX), and the angle BCD is measured by half the arc BAED (B. III, P. XVIII); but the sum of the arcs BCD and AE is equal to the arc BAED, since each is the sum of three arcs subtended by three sides of the pentagon; hence the angle BFD is equal to the angle BCD. In the quadrilateral BCDF, the sum of the four angles is equal to four right angles (B. I, P. XXVI, C. 1), and as the opposite angles B and D are equal, and also C and F, the sum of the angles D and C is equal to two right angles and hence BC and FD are parallel (B. I, P. XIX), and, for a like reason, BF and CD are parallel; hence the figure BCDF is a parallelogram, and the opposite sides, BF and CD, and DF and CB, are equal (B. I, P. XXVIII); also, since BC and CD, being sides of a regular pentagon, are equal to each other, BF is equal to FD; hence the greater segments are equal to each other and to a side of the pentagon; *which was to be proved* in the first case.

The angle AFE equals the angle BFD (B. I, P. II), and hence equals the angle C, one of the angles of the pentagon.

The two triangles AED and AFE have the angles AED (an angle of the pentagon) and AFE equal from what has just been shown, and the angle DAE common, hence they are similar (B. IV, P. XVIII, C.), and have their homologous sides proportional; hence,

$$DA : AE :: DE : EF;$$

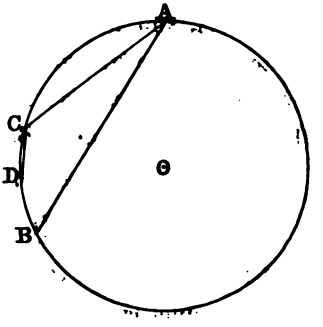
but AE and DE are each equal to DF by the first part of this proposition, and because the angles FAE and FEA, being measured by half the equal arcs ED and AB, are equal, the side EF equals FA (B. I, P. XII); substituting, therefore, we have

$$DA : DF :: DF : FA.$$

In like manner, $BE : BF :: BF : FE;$

hence the diagonals cut each other in extreme and mean ratio; *which was to be proved* in the second case.

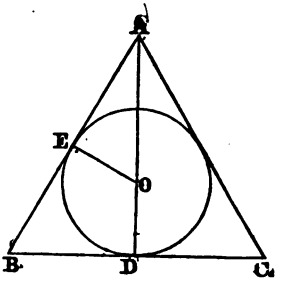
11.—*Show how to inscribe in a given circle a regular polygon of 15 sides.*



Let O be the centre of a given circle. From any point of the circumference, as A, draw AB, the side of an inscribed equilateral triangle (P. V, C. 1), and AC the side of a regular inscribed pentagon (P. VI, C. 2); then since the arc AB, subtended by a side of an inscribed equilateral triangle, is $\frac{1}{3}$ the circumference, and the arc AC, subtended by the side of a regular inscribed pentagon, is $\frac{1}{5}$ the circumference, their differ-

ence, CB, must be equal to $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ of the circumference; hence, the arc CD, one-half of CB, is $\frac{1}{15}$ of the circumference, and hence the chord CD is one side of a regular inscribed polygon of 15 sides; apply CD 15 times as a chord, and the resulting figure will be the polygon required.

12.—*Find the side and the altitude of an equilateral triangle in terms of the radius of the inscribed circle.*



Let ABC be an equilateral triangle, and O the centre of the inscribed circle. Draw the radius OE to the point of contact E, it will be perpendicular to AB (B. III, P. IX). Since the sides AB and AC are equal, the perpendicular AD bisects the base BC (B. I, P. XI, C. 2), and since it also bisects the angle A, it passes through the centre O of the inscribed circle (B. III, Prob. XV); BD = BE (B. III, Prob. XIV, C.), and hence BE, or EA, is one-half the side of the triangle. The triangles ADB and AEO have the angles D and E equal, since each is a right angle, and the angle A common, hence they are similar (B. IV, P. XVIII, C.), and we have

$$AD : DB :: AE : EO;$$

$$\therefore\ AD \times EO = DB \times AE; \quad \ldots\ (1)$$

but DB and AE are equal from what has just been shown, and their product is equal to $\overline{DB}^2$, or to $\frac{1}{4}\overline{BC}^2$ (B. IV, P. VIII, C.), which equals $\frac{1}{4}\overline{AB}^2$. In the triangle ADB,

$$\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2 = \overline{AB}^2 - \tfrac{1}{4}\overline{AB}^2 = \tfrac{3}{4}\overline{AB}^2;$$

hence, $$AD = \sqrt{\tfrac{3}{4}\overline{AB}^2} = \tfrac{1}{2}AB\sqrt{3}. \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (1),

$$\tfrac{1}{2}AB\sqrt{3} \times EO = \tfrac{1}{4}\overline{AB}^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Dividing (3) by $\frac{1}{2}AB$ and calling EO, r, we have

$$r\sqrt{3} = \tfrac{1}{2}AB; \quad \therefore AB = 2r\sqrt{3}; \quad . \quad . \quad . \quad (4)$$

which is the required value of the side.

In equation (2), substituting the value of AB from (4), we have

$$AD = 3r;$$

which is the required value of the altitude.

13.—*Given an equilateral triangle inscribed in a circle, and a similar circumscribed triangle; determine the ratio of the two triangles to each other.*

The two triangles having the same number of sides are similar (P. I), and hence they are to each other as the squares of their homologous sides (B. IV, P. XXV). Letting r denote the radius of the circle, the side of an inscribed equilateral triangle is equal to $r\sqrt{3}$ (P. V, C. 2), and the side of the circumscribed equilateral triangle, as just shown in the preceding exercise, is equal to $2r\sqrt{3}$; hence the circumscribed triangle : the inscribed triangle :: $12r^2 : 3r^2$, or as $4 : 1$; hence the circumscribed equilateral triangle is 4 times the similar inscribed triangle.

14.—*The diameter of a circle is 20 feet; find the area of a sector whose arc is 120°.*

Let S represent the area of the sector, C the area of the

circle, and r the radius; then since there are 360° in the circumference of the circle (see Note, B. III, p. 92*), we have

$$S : C :: 120° : 360° \text{ (P. XIV, C. 2)}$$

hence, $$S = \frac{120}{360}C = \frac{1}{3}C.$$

But from P. XV,

$$C = \pi r^2 = 3.1416 \times 100 = 314.16;$$

hence, $S = \frac{1}{3} \times 314.16 = 104.72$ square feet;

which is the area required.

15.—*The circumference of a circle is 200 feet; find its area.*

Since the circumference equals 3.1416 times the diameter (P. XVI), the diameter equals the circumference divided by 3.1416; hence the diameter of this circle equals

$$\frac{200}{3.1416} = 63.66,$$

and the radius being half the diameter is equal to

$$\frac{63.66}{2} = 31.83,$$

hence, from P. XIV, the area is equal to

$$\frac{200 \times 31.83}{2} = 3183 \text{ square feet.}$$

16.—*The area of a circle is 78.54 square yards; find its diameter.*

Let r denote the radius; then from P. XV,

$$78.54 = \pi r^2;$$

hence, $$r^2 = \frac{78.54}{3.1416} = 25;$$

and, therefore, $$r = 5;$$

but the diameter equals twice the radius, hence the diameter is 10 yards.

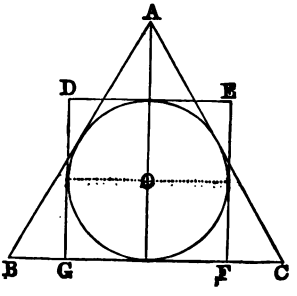
17.—*The radius of a circle is 10 feet and the area of a circular sector 100 square feet; find the arc of the sector in degrees.*

The area of the circle, from P. XV, is

$$3.1416 \times 100 = 314.16 \text{ square feet};$$

hence the sector is $\frac{100}{314.16}$ of the circle; and there being 360° in the circumference of the circle, the arc of the sector is $\frac{100}{314.16}$ of 360° (P. XIV, C. 2); that is, the arc is equal to 114°.588.

18.—*Show that the area of an equilateral triangle circumscribed about a circle is greater than that of a square circumscribed about the same circle.*



Let O be the centre of the circle, ABC the circumscribed equilateral triangle, and DEFG the circumscribed square; and let r denote the radius of the circle.

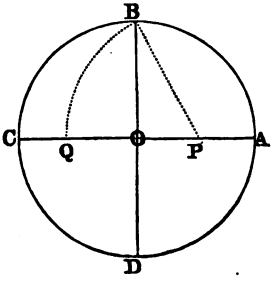
Since the side of the square is equal to the diameter of the circle,

or $2r$, its area is equal to $4r^2$. Since the side of the equilateral triangle is equal to $2r\sqrt{3}$, and the altitude is equal to $3r$ (E. 12, Key), the area of the triangle (B. IV, P. VI) is equal to

$$\frac{2r\sqrt{3} \times 3r}{2} = 3r^2\sqrt{3} = 3r^2 \times 1.732 = 5.196r^2;$$

hence the area of the triangle is greater than that of the square; *which was to be proved.*

19.—*Let AC and BD be diameters perpendicular to each other; from P, the middle point of the radius OA, as a centre, and a radius equal to PB, describe an arc cutting OC in Q; show that the radius OC is divided in extreme and mean ratio at Q.*



It is required to show that

$$OC : OQ :: OQ : QC,$$

or denoting radius OC by r, that

$$\overline{OQ}^2 = r \times QC.$$

In the right-angled triangle POB (from B. IV, P. XI),

$$\overline{PB}^2 = \overline{BO}^2 + \overline{OP}^2 = r^2 + \tfrac{1}{4}r^2 = \tfrac{5}{4}r^2;$$

hence, $$PB = \tfrac{1}{2}r\sqrt{5};$$

but $$PQ = PB,$$

since each is a radius of the arc BQ, and therefore

$$PQ = \tfrac{1}{2}r\sqrt{5}.$$

$$OQ = PQ - PO = \tfrac{1}{2}r\sqrt{5} - \tfrac{1}{2}r = \tfrac{1}{2}r(\sqrt{5} - 1),$$

and hence,

$$\overline{OQ}^2 = \tfrac{1}{4}r^2(\sqrt{5}-1)^2 = \tfrac{1}{4}r^2(5-2\sqrt{5}+1) = \tfrac{1}{4}r^2(6-2\sqrt{5}),$$

or, by factoring the last expression,

$$\overline{OQ}^2 = \tfrac{1}{2}r^2(3-\sqrt{5}). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again,

$$QC = OC - OQ = r - (\tfrac{1}{2}r\sqrt{5} - \tfrac{1}{2}r) = \tfrac{3}{2}r - \tfrac{1}{2}r\sqrt{5};$$

whence, by factoring, $QC = \tfrac{1}{2}r(3-\sqrt{5})$;

multiplying both members of this equation by r, we have

$$r \times QC = \tfrac{1}{2}r^2(3-\sqrt{5}), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since the second members of (1) and (2) are equal, we have

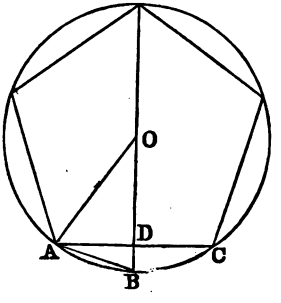
$$\overline{OQ}^2 = r \times QC. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (3), we have the proportion

$$r \text{ or } OC : OQ :: OQ : QC \text{ (B. II, P. II)};$$

hence the radius OC is divided in extreme and mean ratio at Q; *which was to be proved.*

20.—*Show that the square of the side of a regular inscribed pentagon is equal to the square of the side of a regular inscribed decagon, increased by the square of the radius of the circumscribing circle.*



Let AC be the side of a regular pentagon inscribed in a circle whose centre is O, and AB the side of a regular inscribed decagon; denote

AC by p, AB by d, and OA by r; then we are to show that

$$p^2 = d^2 + r^2.$$

Since the side of a regular inscribed decagon is equal to the greater segment of the radius divided in extreme ratio, we shall have the proportion

$$r : d :: d : r - d;$$

whence (B. II, P. I), $r^2 - dr = d^2$;

and by transposition, $d^2 + dr = r^2$; . . . . . (1)

solving equation (1), we have

$$d = -\frac{r}{2} + \sqrt{r^2 + \frac{r^2}{4}} = -\frac{r}{2} + \frac{r}{2}\sqrt{5}; \quad . \quad . \quad (2)$$

squaring both members of (2), we have

$$d^2 = \frac{r^2}{4} - \frac{r^2}{2}\sqrt{5} + \frac{5r^2}{4} = \frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5}; \quad . \quad . \quad (3)$$

and adding r^2 to both members of (3),

$$d^2 + r^2 = \frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5} + r^2 = \frac{5r^2}{2} - \frac{r^2}{2}\sqrt{5}. \quad . \quad . \quad (4)$$

Again, $\overline{AD} = \frac{1}{2}AC$ (B. III, P. VI),

and hence $\overline{AD}^2 = \frac{1}{4}p^2$.

In the right-angled triangle ODA,

$$\overline{OD}^2 = \overline{OA}^2 - \overline{AD}^2 = r^2 - \frac{1}{4}p^2;$$

and hence $OD = \sqrt{r^2 - \frac{1}{4}p^2}$;

$$DB = OB - OD = r - \sqrt{r^2 - \frac{1}{4}p^2},$$

whence by squaring,

$$\overline{DB}^2 = r^2 - 2r\sqrt{r^2 - \tfrac{1}{4}p^2} + r^2 - \tfrac{1}{4}p^2 = 2r^2 - 2r\sqrt{r^2 - \tfrac{1}{4}p^2} - \tfrac{1}{4}p^2.$$

In the right-angled triangle ADB, we have

$$\overline{AB}^2, \text{ or } d^2 = AD^2 + DB^2;$$

whence, by substituting for $\overline{AD}^2$ and $\overline{DB}^2$ their values found above, we have

$$d^2 = \tfrac{1}{4}p^2 + 2r^2 - 2r\sqrt{r^2 - \tfrac{1}{4}p^2} - \tfrac{1}{4}p^2 = 2r^2 - 2r\sqrt{r^2 - \tfrac{1}{4}p^2};$$

whence, by transposition,

$$2r\sqrt{r^2 - \tfrac{1}{4}p^2} = 2r^2 - d^2;$$

squaring both members of this equation, we have

$$4r^4 - p^2r^2 = 4r^4 - 4r^2d^2 + d^4;$$

transposing and reducing,

$$p^2r^2 = 4r^2d^2 - d^4 = d^2(4r^2 - d^2).$$

Substituting for d^2 its value from equation (3), we have

$$p^2r^2 = \left(\frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5}\right)\left[4r^2 - \left(\frac{3r^2}{2} - \frac{r^2}{2}\sqrt{5}\right)\right];$$

performing indicated operations and reducing,

$$p^2r^2 = \frac{5r^4}{2} - \frac{r^4}{2}\sqrt{5},$$

dividing both members of this by r^2, we have

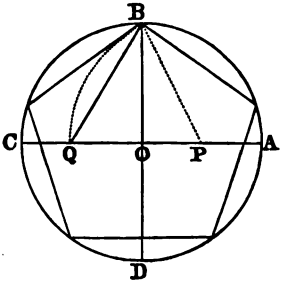
$$p^2 = \frac{5r^2}{2} - \frac{r^2}{2}\sqrt{5}. \quad \ldots\ldots \quad (5)$$

Since the second members of equations (4) and (5) are equal, the first members are equal, and we have

$$p^2 = d^2 + r^2;$$

which was to be proved.

21.—*Show how, from 19 and 20, to inscribe a regular pentagon in a given circle.*



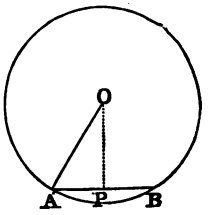
Let ABCD be the given circle. Draw two diameters, AC and BD, perpendicular to each other, with P, the middle point of OA, as a centre, and a radius equal to PB, describe the arc BQ, cutting the radius OC in Q; then is the radius OC divided in extreme and mean ratio at Q (E. 19), and OQ, the greater segment, is equal to the side of a regular inscribed decagon (P. VI); draw BQ, and in the right-angled triangle BOQ,

$$\overline{BQ}^2 = \overline{OQ}^2 + \overline{OB}^2 \text{ (B. IV, P. XI)};$$

but OQ being the side of a regular inscribed decagon, and OB the radius of the circumscribed circle, BQ must be equal to the side of a regular inscribed pentagon (E. 20). Beginning at any point of the circumference, as B, apply BQ five times as a chord; the resulting figure will be the pentagon required.

22.—*The side of a regular pentagon, inscribed in a circle, is 5 feet, and that of a regular inscribed decagon is 2.65 feet; find the side and the area of a regular hexagon inscribed in the same circle.*

Letting p denote a side of the pentagon, d a side of the decagon, and r the radius of the circle, we have



$$p^2 = d^2 + r^2 \text{ (E. 20)};$$

but the side of a regular inscribed hexagon is equal to the radius of the circle (P. IV), hence calling the side of the hexagon h, we have

$$p^2 = d^2 + h^2;$$

whence

$$h^2 = p^2 - d^2.$$

and substituting the given numerical values, we have

$$h^2 = 25 - 7.02 = 17.98;$$

hence,

$$h = 4.24.$$

In the circle whose centre is O, let AB be the side of a regular hexagon, OA the radius, and OP the apothem of the hexagon. In the right-angled triangle OPA, we have

$$\overline{OP}^2 = \overline{OA}^2 - \overline{AP}^2;$$

but

$$AP = \tfrac{1}{2}AB \text{ (B. III, P. VI)};$$

or since

$$AB = OA,$$

$$AP = \tfrac{1}{2}OA,$$

and

$$\overline{AP}^2 = \tfrac{1}{4}\overline{OA}^2;$$

hence, $\overline{OP}^2 = \overline{OA}^2 - \frac{1}{4}\overline{OA}^2 = \frac{3}{4}OA^2$.

But OA, being equal to AB, is equal to 4.24; hence

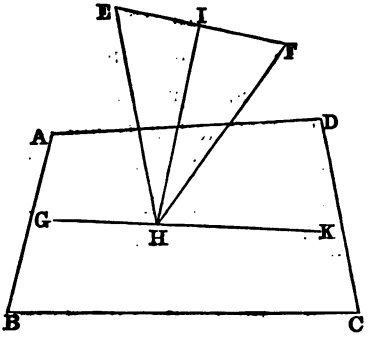
$$\overline{OP}^2 = \frac{3}{4} \times 17.98 = 13.49;$$

and, therefore, $OP = \sqrt{13.49} = 3.67$.

The area of the hexagon is equal to half the product of its perimeter and apothem (P. VIII), the perimeter is equal to one side multiplied by the number of sides; *i. e.*, to $4.24 \times 6 = 25.44$; the apothem $= 3.67$, as just found; hence, the area $= \frac{25.44 \times 3.67}{2} = 46.6824$ square feet; which was to be found.

BOOK VI.

1.—*Find a point in a plane equidistant from two given points without and on the same side of the plane.*



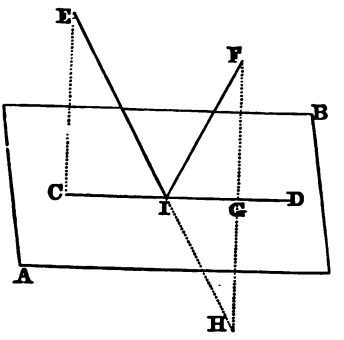
Let E and F be the given points, and ABCD the given plane. Draw the line EF and through it pass a plane cutting the given plane in the line GK; in this plane draw the line IH perpendicular to EF at its middle point, and let it meet the given plane at the point H; draw EH and FH, and they will be equal (B. I, P. XVI); hence H is the point required.

Cor.—As an infinite number of planes can be drawn through EF cutting the plane AC, an infinite number of points can be found to satisfy the conditions.

2.—*From two given points on the same side of a given plane, draw two lines that shall meet the plane and make equal angles with it.*

Let E and F be the given points and AB the given plane. Through the given points pass a plane perpendicular to the

given plane, and cutting it in the line CD; in this plane draw FG perpendicular to CD, it will also be perpendicular to the plane AB (P. XVII); prolong FG till GH equals FG; draw HE, and to the point I, in which this line pierces the plane AB, draw FI; EI and FI are the lines required: for the triangles FGI and HGI are equal in all their parts (B. I, P. V), and hence the angles HIG and FIG are equal to each other; but angle HIG equals angle EIC (B. I, P. II), and, therefore, the angles EIC and FIG are equal, and these are the angles which the lines EI and FI make with the plane.

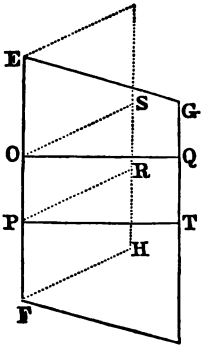


3.—*What is the greatest number of equilateral triangles that can be grouped about a point so as to form a convex polyedral angle?*

Five. For each angle of an equilateral triangle is $\frac{2}{3}$ of a right angle, and five of them equal $\frac{10}{3}$ of a right angle; six such angles would be $\frac{12}{3}$ of a right angle, or 4 right angles, which exceeds the possible sum of the plane angles formed by the edges of any convex polyedral angle (P. XX); hence no more than 5 such triangles can be grouped so as to form a convex polyedral angle.

4.—*Show that if from any two points in the edge of a diedral angle straight lines be drawn in each of its faces perpendicular to the edge, these lines will contain equal angles.*

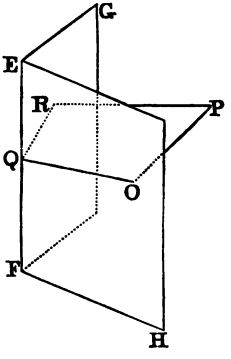
In the diedral angle whose edge is EF, draw from the points O and P in the edge, the lines OS and PR in the face EH, and the lines OQ and PT in the face FG, perpendicular to the edge EF, then will the angles QOS and TPR, contained by these lines, be equal.



Since OS and PR are in the plane EH, and both perpendicular to the same line EF, they are parallel (B. I, P. XVIII); for a like reason OQ and PT are parallel; hence the angles QOS and TPR are equal (P. XIII); *which was to be proved.*

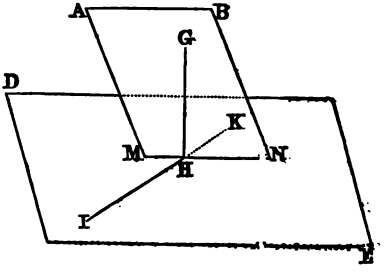
5.—*From any point within a diedral angle, draw a perpendicular to each of its two faces, and show that the angle contained by the perpendiculars is the supplement of the diedral angle.*

From a point P within the diedral angle whose edge is EF, draw PR and PO perpendicular, respectively, to the faces FG and EH; then will the angle RPO be the supplement of the diedral angle. Through the lines PR and PO pass a plane, and let its intersections with the faces FG and EH be RQ and OQ; the edge EF is perpendicular to the plane of the lines PR and PO (P. XVIII), and hence the plane angle RQO is equal to the given diedral angle (D. 4). In the quadrilateral PRQO, the angles PRQ and POQ are right angles by hypothesis; and since the sum of all the



interior angles of the quadrilateral is equal to 4 right angles (B. I, P. XXVI, C. 1), the sum of the remaining angles, RPO and RQO, is equal to two right angles, and hence RPO is the supplement of the angle RQO (see Note 2, p. 49**), and therefore of the given diedral angle; *which was to be proved.*

6.—*Show that if a plane meet another plane, the sum of the adjacent diedral angles will be equal to two right angles.*

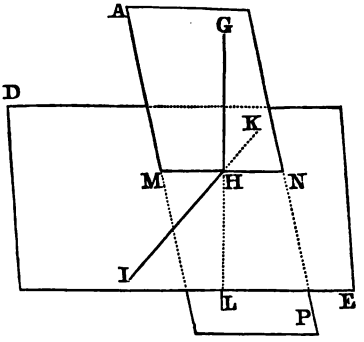


Let the plane AN meet the plane DE in the line MN. In the plane AN, draw the line GH perpendicular to the line of intersection MN, and in the plane DG, draw IH perpendicular to MN and prolong it to K; the plane angle GHI is equal to the diedral angle formed by the planes (D. 4), and GHK is equal to the adjacent diedral angle; since the sum of the plane angles GHI and GHK is equal to two right angles (B. I, P. I), the sum of the adjacent diedral angles is equal to two right angles; *which was to be proved.*

7.—*Show that if two planes intersect each other, the opposite or vertical diedral angles will be equal to each other.*

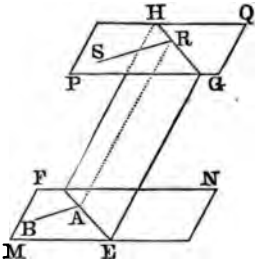
Let the planes AP and DE intersect each other in the line MN. In the plane AP, draw GH perpendicular to the intersection MN at H, and prolong it to L; in the plane DE, draw IH perpendicular to MN at H, and prolong it to K; then the plane

angle GHI is equal to the diedral angle formed by the planes, and GHI and LHK, or GHK and IHL, are equal to the vertical diedral angles formed by the planes (D. 4); since these vertical plane angles are equal to each other (B. I, P. II), their equals, the vertical diedral angles, are equal to each other; *which was to be proved.*



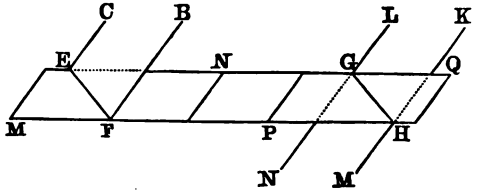
8.—*Show that if a plane intersect two parallel planes, the sum of the interior diedral angles on the same side will be equal to two right angles.*

Let the plane HE intersect the parallel planes PQ and MN. In the plane HE draw the line AR perpendicular to the line HG, the intersection of the planes HE and PQ; it will also be perpendicular to FE, the intersection of the planes HE and MN (B. I, P. XX, C. 1); in the plane PQ, draw RS perpendicular to HG, and in the plane MN, draw AB perpendicular to FE; the plane angles SRA and BAR are equal respectively to the diedral angles formed by the planes HE and PQ, and the planes HE and MN (D. 4); the lines RS and AB are parallel (B. I, P. XVIII), and hence the sum of the angles SRA and RAB is equal to two right angles (B. I, P. XX), and hence



the sum of their equals, the interior diedral angles on the same side, will be equal to two right angles; *which was to be proved.*

9.—*Show that if two diedral angles have their faces parallel and lying in the same or in opposite directions, they will be equal.*



1°. Let the diedral angles LGH–Q and CFE–N have their faces LGHK and CEFB, and GHQ and EFN, parallel and lying in the same direction; then will they be equal.

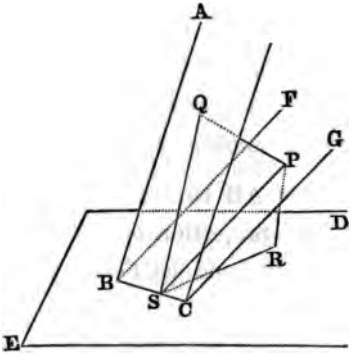
For place the planes MN and PQ so that they form one and the same plane; then, since the planes LH and CF are parallel and are met by the plane MQ, the sum of the interior diedral angles on the same side, LGH–P and CEF–N, is equal to two right angles (E. 8); and since the plane LH meets the plane PQ, the sum of the adjacent diedral angles, LGH–P and LGH–Q, is equal to two right angles (E. 6); hence, the sum of LGH–P and CEF–N is equal to the sum of LGH–P and LGH–Q (A. 1), and taking away the common diedral angle LGH–P, we have the angle CEF–N equal to the angle LGH–Q; *which was to be proved* in the first case.

2°. Let the diedral angles CEF–N and MHG–P, have their faces parallel and lying in opposite directions, then will they be equal.

For let the planes MN and PQ form one and the same plane, as before; prolong the plane MG; then since the planes ML

and PQ intersect, the vertical diedral angles MHG–P and LGH–Q are equal (E. 7); but LGH–Q and CEF–N have their faces parallel and lying in the same direction, and are equal as just shown; and, therefore, MHG–P and CEF–N are equal; *which was to be proved* in the second case.

10.—*Show that every point of a plane bisecting a diedral angle is equidistant from the faces of the angle.*



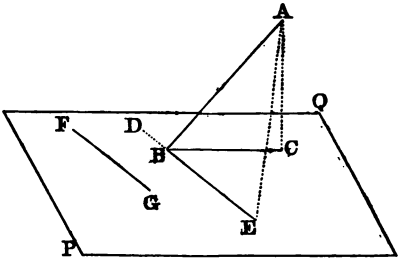
Let the diedral angle ABC–D be bisected by the plane FBC–G, then will the plane be everywhere equidistant from the faces AC and ED.

For take any point, as P, of the plane, and draw PQ perpendicular to the plane AC, and PR perpendicular to the plane ED; through the lines PQ and PR pass a plane and let it intersect the planes AC, FC, and ED, respectively, in the lines QS, PS, and RS; the plane PQSR will be perpendicular to the planes AC and ED (P. XVI), and hence the line BC is also perpendicular to the plane PQSR (P. XVIII), and hence is perpendicular to the lines SQ, SP, and SR, in that plane (D. 1); therefore the plane angle QSR is equal to the diedral angle ABC–D (D. 4), and the plane angles QSP and PSR are equal, respectively, to the diedral angles ABC–F and FBC–D; but these last diedral angles are equal by hypothesis, and, consequently, the plane angles

QSP and PSR are equal, and hence the line PS bisects the angle QSR; then the plane triangles PQS and PRS have the angles PQS and PRS equal (A. 10), the angles PSQ and PSR equal, as just shown, and the side PS common, hence they are equal in all their parts, and PQ equals PR; hence the point P is equally distant from the planes AC and ED; *which was to be proved.*

Cor.—Any point, as P, which is equally distant from the faces of a diedral angle, lies in the plane which bisects the diedral angle.

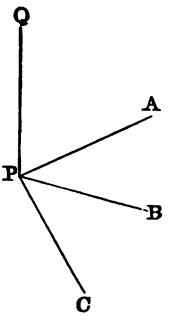
11.—*Show that the inclination of a line to a plane—that is, the angle which the line makes with its own projection on the plane—is the least angle made by the line with any line of the plane.*



Let AB be a line and BC its projection on the plane PQ, and let FG be any line of the plane other than BC; then will AB make a less angle with BC than with FG.

For through B draw in the plane PQ the line DE parallel to FG; then is the angle ABE equal to the angle made by AB with FG (P. VI, S.); lay off BE equal to BC, and draw AE; then the triangles ABC and ABE have the sides BE and BC equal, and the side AB common; but the side AC is less than the side AE (P. V), hence the angle ABC is less than the angle ABE (B. I, P. IX), and, therefore, is the least angle made by the line AB with any line of the plane; *which was to be proved.*

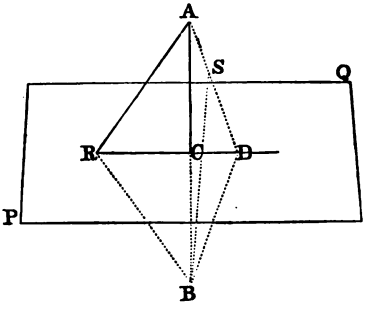
12.—*Show that if three lines are perpendicular to a fourth at the same point, the first three will be in the same plane.*



Let the three lines AP, BP, and CP be perpendicular to the line PQ at the point P; then will AP, BP, and CP, be in the same plane.

For pass a plane through the lines AP and BP, it will be perpendicular to the line PQ (P. IV); in like manner pass a plane through the lines BP and CP, and it will also be perpendicular to the line PQ, and hence these two planes must be parallel to each other (P. IX); and since they have a common point P, they must coincide and form one and the same plane; *which was to be proved.*

13.—*Show that when a plane is perpendicular to a given line at its middle point, every point of the plane is equally distant from the extremities of the line, and that every point out of the plane is unequally distant from the extremities of the line.*



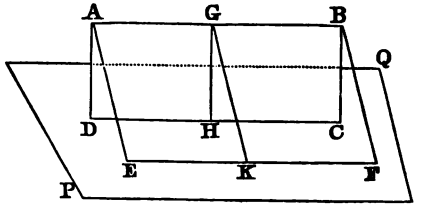
Let the plane PQ be perpendicular to the line AB at its middle point C, and let R be any point of the plane; then is R equally distant from A and B.

For through R and AB pass a plane, it will intersect the plane PQ in the line RC;

draw the lines AR and BR; the two triangles ACR and BCR have the angles at C equal, each being a right angle by hypothesis; the side AC equal to the side CB by hypothesis, and the side RC common; they are, therefore, equal in all their parts, and hence AR equals BR; *which was to be proved* in the first case.

Let S be a point without the plane, then is S unequally distant from A and B. For through S and AB pass a plane, its intersection with PQ will be CD; draw SB and SA; prolong AS to D and draw DB. DB is equal to DA as just shown. DB — DS $<$ SB (B. I, P. VII, C.), and, therefore, DA — DS, or SA $<$ SB; hence S is unequally distant from A and B; *which was to be proved* in the second case.

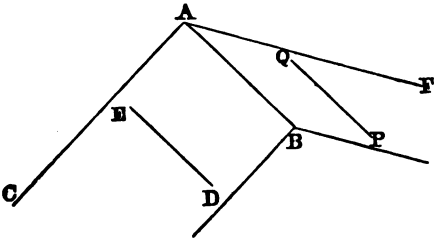
14.—*Show that through a line parallel to a given plane, but one plane can be passed perpendicular to the given plane.*



Let AB be parallel to the plane PQ. Through AB pass the plane ABCD, perpendicular to PQ, and let it intersect PQ in the line CD. If another plane can be passed through AB perpendicular to PQ, let AEFB be that plane. From G, any point of the line AB, draw in the plane AC, the line GH perpendicular to the line CD, then will GH be perpendicular to the plane PQ (P. XVII); from the same point G, draw in the plane AF the line GK perpendicular to the line EF, then will GK be perpendicular to the plane PQ (P. XVII); hence from G two perpendiculars have been drawn to the plane

PQ, which is impossible (P. IV, C. 1); hence but one plane can be passed through AB perpendicular to the plane PQ; *which was to be proved.*

15.—*Show that if two planes which intersect contain two lines parallel to each other, the intersection of the planes will be parallel to the lines.*



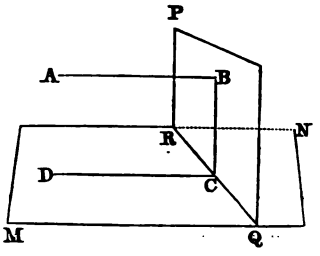
Let the planes BF and BC intersect, and let PQ in the plane BF be parallel to DE in the plane BC; then will AB, the intersection of the planes, be parallel to DE and PQ.

Since PQ is parallel to the line DE of the plane BC, it is parallel to the plane BC (P. VIII), and cannot, therefore, meet the line AB, which lies in that plane; but AB lies, also, in the plane BF, and since AB and PQ lie in the same plane and cannot meet, how far soever they may be produced, they are parallel; and, further, since AB and DE are parallel to PQ, they are parallel to each other (P. VII, C. 2); hence AB is parallel to the lines PQ and DE; *which was to be proved.*

16.—*Show that when a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.*

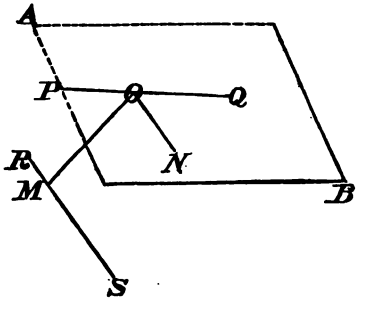
Let the line AB be parallel to the plane MN, and perpendicular to the plane PQ; then will the planes AB and PQ be perpendicular to each other.

For in the plane PQ, draw BC perpendicular to QR; through



the lines AB and BC pass a plane; CD will be its intersection with the plane MN; AB is parallel to CD, for if it could meet CD at any point, it would meet the plane MN in which CD lies, which is impossible, since AB is parallel to the plane MN by hypothesis; because AB is perpendicular to the plane PQ, DC is also perpendicular to PQ (P. VII), and is, therefore, perpendicular to the lines RQ and CB; therefore, since BC in the plane PQ, and DC in the plane MN, are perpendicular to RQ at the same point C, the angle BCD is equal to the angle formed by the planes PQ and MN; and since this angle is a right angle, the planes are perpendicular to each other; *which was to be proved.*

17.—*Draw a perpendicular to two lines not in the same plane.*

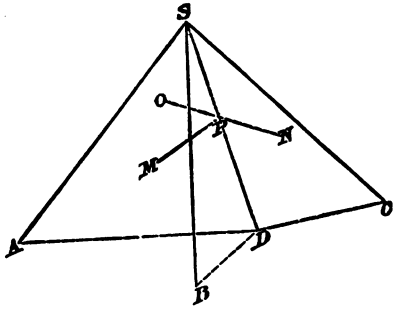


Let PQ and RS be two lines not in the same plane; it is required to draw a line perpendicular to both of them.

Through any point of PQ, as O, draw ON parallel to RS; through PQ and ON pass the plane AB; then since RS is parallel to ON, it is parallel to the plane AB (P. VIII);

at O, draw OM perpendicular to the plane AB (P. V, C,); then since OM is perpendicular to ON, it is perpendicular to its parallel RS (B. I, P. XX, C. 1); and being perpendicular to the plane AB, is perpendicular to the line PQ also, and is the perpendicular required.

18.—*Show that the three planes which bisect the diedral angles formed by the consecutive faces of a triedral angle, meet in the same line.*



Let S be the vertex of a triedral angle, and let the diedral angle ABS–C be bisected by the plane BSD, the diedral angle BCS–A be bisected by the plane CSD, and the diedral angle CAS–B be bisected by the plane ASD; then will the planes BSD, CSD, and ASD, meet in the same line. Let the two planes BSD and CSD meet in the line SD (P. III) take any point, as P, of this line, and draw from it PM, PN, and PO, perpendicular, respectively, to the planes ASB, BSC, and CSA; then since the plane BSD bisects the diedral angle ABS–C, PM and PN are equal (E. 10); and since the plane CSD bisects the diedral angle BCS–A, PN and PO are equal (E. 10); hence, PM and PO are equal, and, therefore, P is equally distant from the faces of the diedral angle CAS–B, and, consequently, lies in the

plane which bisects that angle (E. 10, C.); but P is any point of the line SD, and, therefore, every point of SD lies in the plane which bisects the diedral angle CAS–B; hence the three bisecting planes meet in the same line SD; *which was to be proved.*

BOOK VII.

1.—*What is the convex surface of a right prism whose altitude is 20 feet and whose base is a pentagon each side of which is 15 feet?*

The perimeter of the base is

$$15 \times 5 = 75,$$

and the convex surface is (P. I) equal to

$$75 \times 20 = 1500 \text{ square feet.}$$

2.—*The altitude of a pyramid is 10 feet and the area of its base is 25 square feet; find the area of a section made by a plane 6 feet from the vertex and parallel to the base.*

Let x represent the area of the section required. Then (P. III, C. 3),

$$x : 25 :: 36 : 100;$$

hence,
$$x = \frac{25 \times 36}{100} = 9 \text{ square feet.}$$

3.—*Find the convex surface of a right triangular pyramid, each side of the base being 4 feet and the slant height 12 feet.*

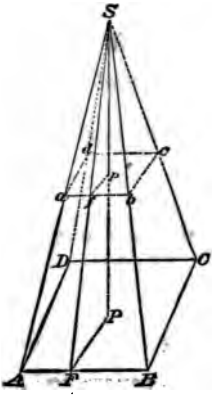
The perimeter of the base is

$$4 \times 3 = 12 \text{ feet,}$$

and the convex surface is (P. IV) equal to

$$12 \times 12 = 72 \text{ square feet.}$$

4. *A right pyramid whose altitude is 8 feet and whose base is a square each side of which is 4 feet, is cut by a plane parallel to the base and 2 feet from the vertex; required the convex surface of the frustum included between the base and the cutting plane.*



Let S–ABCD be the right pyramid, *abcd* a section 2 feet from the vertex, SP the altitude. Draw the slant height SF, and connect the points P and F. Since ABCD is a square and P is the centre of the base (D. 11) the line PF is equal to one-half the side of the base, or 2; then in the right-angled triangle SPF,

$$\overline{SF}^2 = \overline{SP}^2 + \overline{PF}^2 \text{ (B. IV, P. XI);}$$

that is, $\overline{SF}^2 = 64 + 4 = 68.$

hence, $SF = \sqrt{68} = 8.25$

nearly. Since the section *abcd* is parallel to the base, SP is perpendicular to *abcd* (B. VI, P. XI); and hence the triangles SPF and S*pf* have the right angles SPF and S*pf* equal and the angle at S common, hence they are similar (B. IV, P. XVIII, C.) and we have

$$SP : Sp :: SF : Sf,$$

or

$$8 : 2 :: 8.25 : Sf;$$

hence,

$$Sf = \frac{2 \times 8.25}{8} = 2.06.$$

Since SF equals 8.25 and $Sf = 2.06$, their difference fF, or the slant height of the frustum, equals 6.19.

Again, since *abcd* is parallel to the base, we have (P. III, C. 3)

$$abcd : \text{ABCD} :: \overline{Sp}^2 : \overline{SP}^2,$$

or $$abcd : \text{ABCD} :: 4 : 64;$$

hence, $$abcd = \tfrac{1}{16}\,\text{ABCD} = 1;$$

and since *abcd* is similar to the base (P. III) it is a square, and its area being 1, each of its sides must be 1. Hence the perimeter of the upper base of the frustum is 4, the perimeter of the lower base 16, and the slant height 6.19, and therefore its convex surface is equal to (P. IV, S.)

$$\frac{4+16}{2} \times 6.19 = 61.9 \text{ square feet};$$

which is the area required.

5.—*The three concurrent edges of a rectangular parallelopipedon are 4, 6, and 8 feet; find the length of the diagonal.*

The square of the diagonal is equal to the sum of the squares of 4, 6, and 8 (P. VI, C. 2), that is, to

$$16 + 36 + 64 = 116;$$

hence the diagonal is the square root of 116, *i. e.*, 10.77.

6.—*Of two rectangular parallelopipedons having equal bases, the altitude of the first is 12 feet and its volume is 275 cubic feet; the altitude of the second is 8 feet—find its volume.*

Let V denote the volume required; then (P. XIV, C.)

$$V : 275 :: 8 : 12;$$

hence, $V = \frac{275 \times 8}{12} = 183\frac{1}{3}$ cubic feet.

7.—*Two rectangular parallelopipedons having equal altitudes are, respectively, 80 and 45 cubic feet in volume, and the area of the base of the first is 12 square feet; find the base of the second and the altitude of both.*

Since the volume is equal to the base multiplied by the altitude (P. XIII, C. 2), the altitude must be equal to the volume divided by the base; hence the altitude of the first parallelopipedon, must be equal to $\frac{80}{12} = 6\frac{2}{3}$ feet, and since they have equal altitudes, the altitude of the second is, also, $6\frac{2}{3}$ feet; the base of the second is equal to its volume divided by its altitude, that is, to

$$\frac{45}{6\frac{2}{3}} = 6.75 \text{ square feet.}$$

8.—*Find the volume of a triangular prism whose base is an equilateral triangle of which the altitude is 3 feet, the altitude of the prism being 8 feet.*

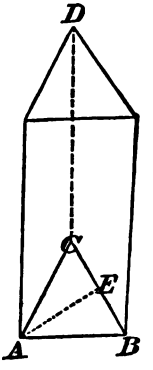
In the base ABC, draw AE to the middle point of BC, then is AE the altitude of ABC (B. I, P. XI, C. 2); BE being one-half of BC or its equal AB,

$$\overline{BE}^2 = \tfrac{1}{4}\,\overline{AB}^2.$$

In the triangle AEB,

$$\overline{AB}^2 - \overline{BE}^2 = \overline{AE}^2 \text{ (B. IV, P. XI, C. 1);}$$

that is, $\overline{AB}^2 - \tfrac{1}{4}\,\overline{AB}^2$,



or $$\tfrac{1}{4}\overline{AB}^2 = 9;$$

hence, $$\overline{AB}^2 = 12,$$

and $$AB = \sqrt{12} = 3.464;$$

therefore the area of the base of the prism is

$$\frac{3.464 \times 3}{2} = 5.196 \text{ square feet,}$$

and the volume of the prism is (P. XIV) equal to

$$5.196 \times 8 = 41.568 \text{ cubic feet.}$$

9.—*The volumes of two pyramids having equal altitudes are, respectively, 60 and 115 cubic yards, and the base of the smaller is 8 square yards; find the base of the larger.*

Let x denote the base required; then (P. XVII, C. 2) we have

$$60 : 115 :: 8 : x;$$

hence, $$x = \frac{115 \times 8}{60} = 15\tfrac{1}{3} \text{ square yards.}$$

10.—*Given a pyramid whose volume is 512 cubic feet and altitude 8 feet; find the volume of a similar pyramid whose altitude is 12 feet, and find, also, the area of the base of each.*

Let V denote the volume of the pyramid required; then, as the volumes are to each other as the cubes of the altitudes (P. XX, C.), we have

$$V : 512 :: 1728 : 512;$$

hence, $$V = 1728 \text{ cubic feet.}$$

Let b denote the base of the first pyramid; then (P. XVII)

$$b \times \tfrac{8}{3} = 512;$$

hence, $$b = 512 \times \tfrac{3}{8} = 192 \text{ square feet,}$$

the area of the base of the first. In like manner if b' denote the base of the second pyramid,

$$b' \times 1\tfrac{2}{3} = 1728;$$

hence, $$b' = 1728 \times \tfrac{3}{1\,2} = 432 \text{ square feet,}$$

the base of the second pyramid.

11.—*Find the volume of the frustum of a right triangular pyramid with each side of the lower base 6 feet and each side of the upper base 4 feet, the altitude being 5 feet.*

The area of the lower base is equal to the square of 6 multiplied by .433 (B. IV, E. 19, Key), that is, to

$$36 \times .433 = 15.588.$$

In like manner, the area of the upper base is equal to

$$16 \times .433 = 6.928.$$

Let x denote the mean proportional between the upper and lower bases, then (B. II, D 4),

$$15.588 : x :: x : 6.928;$$

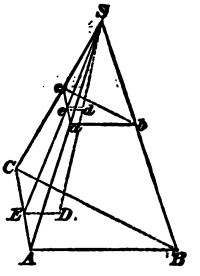
hence, $$x = \sqrt{15.588 \times 6.928} = 10.392.$$

Hence the volume of the frustum is (P. XVIII) equal to

$$(15.588 + 6.928 + 10.392) \times \tfrac{5}{3};$$

which is, 54.846 cubic feet.

12.—*Find the volume of the pyramid of which the frustum given in the last example is a frustum.*



Let ABC–*c* be the frustum given, and let S–ABC be the pyramid whose volume is required. Draw SD, the altitude of the pyramid, and let it pierce the section *abc* in *d*; then are D and *d* the centres of the circles inscribed in ABC and *abc* (D. 11), and DE and *de* the radii of these circles.

As found in the last example, 15.588 is the area of ABC, and 6.928 the area of *abc*; but the area is equal to half the product of the perimeter and radius of the inscribed circle (B. IV, P. VI, C. 2), therefore

$$\frac{DE \times 18}{2} = 15.588,$$

and

$$\frac{de \times 12}{2} = 6.928;$$

hence,

$$DE = 1.732;$$

and

$$de = 1.155 \text{ nearly.}$$

Draw SE, the slant height of the pyramid, then the triangles SDE and S*de* are similar (B. IV, P. XXI), hence

$$SD : Sd :: DE : de;$$

that is,

$$SD : Sd :: 1.732 : 1.155;$$

hence,

$$1.732Sd = 1.155SD,$$

but

$$SD = Sd + dD;$$

that is, since dD is equal to 5 by hypothesis,

$$\text{SD} = \text{S}d + 5;$$

therefore $1.732\text{S}d = 1.155\ (\text{S}d + 5) = 1.155\text{S}d + 5.775;$

transposing $1.155\text{S}d$ to the first member, we have

$$1.732\text{S}d - 1.155\text{S}d = 5.775,$$

$$.577\text{S}d = 5.775;$$

hence, $\text{S}d = 10$ very nearly.

Since Sd is 10 and dD is 5, SD, or the altitude of the pyramid, is equal to their sum, or 15. Hence the volume of S–ABC is equal to

$$\frac{15.588 \times 15}{3} = 77.94 \text{ cubic feet.}$$

13.—*Given two similar prisms; the base of the first contains 30 square yards and its altitude is 8 yards; the altitude of the second prism is 6 yards—find its volume and the area of its base.*

The volume of the first prism is

$$30 \times 8 = 240 \text{ cubic yards (P. XIV);}$$

then (P. XIX, C. 2) we have, representing the volume of the second prism by V,

$$\text{V} : 240 :: 216 : 512;$$

therefore, $\text{V} = \frac{240 \times 216}{512} = 101.25$ cubic yards.

Since the volume is equal to the base multiplied by the alti-

tude, the base must be equal to the volume divided by the altitude; that is, to

$$\frac{101.25}{6} = 16.875.$$

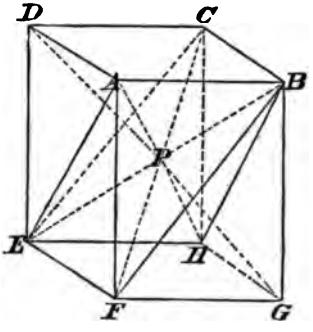
14.—*A pyramid, whose base is a regular pentagon of which the apothem is 3.5 feet, contains 129 cubic feet; find the volume of a similar pyramid, the apothem of whose base is 4 feet.*

Let V denote the volume required; then since the apothems of the bases are homologous lines, we have (P. XX, C.),

$$V : 129 :: 64 : 42.875;$$

hence,
$$V = \frac{129 \times 64}{42.875} = 192.56.$$

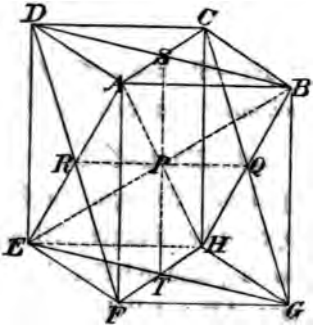
15.—*Show that the four diagonals of a parallelopipedon bisect each other in a common point.*



Let EFGH–C be a parallelopipedon. Through the diagonally opposite basal edges, AB and EH, pass a plane ABHE, its intersections, AE and BH, with the lateral faces AFED and BGHC, will be parallel (B. VI, P. X), and the figure ABHE is a parallelogram, of which the diagonals, AH and BE, bisect each other at P (B. I, P. XXXI). Through the edges BC and EF, pass a plane BCEF; then will BF and CE, its intersections with the lateral faces ABGF and CDEH, be parallel, and the figure BCEF is a

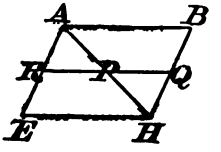
parallelogram, of which the diagonals, CF and BE, bisect each other; and since P is the middle point of BE, as just shown, they bisect each other at P. In like manner, by passing a plane through the edges DC and FG, it may be shown that the diagonals DG and CF bisect each other at P; hence the four diagonals, AH, BE, CF, and DG, bisect each other at P; *which was to be proved.*

16.—*Show that the two lines joining the points of the opposite faces of a parallelopipedon, in which the diagonals of those faces intersect, bisect each other at the point in which the diagonals of the parallelopipedon intersect.*



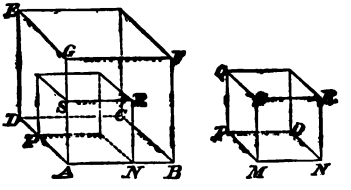
Let ST and RQ join the points of the opposite faces AC and FH, and DF and CG, in which the diagonals of these faces intersect; then will ST and RQ bisect each other at P, the point in which the diagonals of the parallelopipedon intersect. For, R is the middle point of the line AE, and Q is the middle point of the line BH (B. I, P. XXXI); in the parallelogram ABHE, the line RQ, joining the middle points R and Q, of the opposite sides AE and BH, passes through the point P, in which the diagonals, BE and AH, of the parallelogram intersect, and is bisected at that point (see Note at the end). In like manner, in the parallelogram DBGE, the line ST, joining the middle points S and T, of the opposite sides DB and EG, passes through the point in which the diagonals, DG and BE,

of the parallelogram intersect; that is, through the point P (E. 15), and is bisected at that point; hence the lines, RQ and ST, bisect each other at P, the point in which the diagonals of the parallelopipedon intersect; *which was to be proved.*



NOTE.—In the parallelogram ABHE, join the middle points, R and Q, of the opposite sides, AE and BH, and draw the diagonal AH. In the two triangles, APR and QPH, the angles at P are equal (B. I, P. II), the angles PAR and PHQ are equal (B. I, P. XX, C. 2), and, therefore, the angles ARP and PQH are equal (B. I, P. XXV, C. 2); the side AR = QH, by hypothesis; hence the triangles are equal in all their parts (B. I, P. VI), and the side AP = PH, and, consequently, P is the middle point of the diagonal AH, and is the point in which the diagonals of the parallelogram intersect (B. I, P. XXXI).

17.—*Show that two regular polyedrons of the same kind are similar.*



Take, for example, two hexaedrons, or cubes. They have the same number of faces, since each has six. Each face of one is similar to the corresponding face of the other, since they are regular polygons of the same number of sides (B. V, P. I). Place the cube PR on the cube DF, so that the plane angle NMP shall fall upon its equal angle BAD, the line MN will take the direction AB, and MP the direction AD; since the plane angle NMS is equal to BAG, and MN takes the direction AB, the line MS will take the direction AG, and PR will take the position

shown in the figure, in which the corresponding faces of the two polyedrons are like placed; hence they are similar (D. 16). In like manner, any other two regular polyedrons of the same kind may be shown to be similar; *which was to be proved.*

18.—*Show that the surfaces of any two similar polyedrons are to each other as the squares of any two homologous edges.*

Assume the figure in the last exercise; we have

$$DG : PS :: \overline{AG}^2 : \overline{MS}^2 \text{ (B. IV, P. XXVII)};$$

hence, $6DG : 6PS :: AG^2 : \overline{MS}^2$ (B. II, P. VII and IV).

But since the 6 faces of the hexaedron DF are equal, 6DG is equal to the entire surface of DF; for a like reason, 6PS is equal to the entire surface of PR; hence,

$$\text{Surf. DF} : \text{Surf. PR} :: \overline{AG}^2 : \overline{MS}^2.$$

In the same way, the surfaces may be shown to be to each other as the square of any other two homologous edges. It may be shown, in like manner, that the surfaces of any other two similar polyedrons are to each other as the squares of any two homologous edges; *which was to be proved.*

BOOK VIII.

1.—*The radius of the base of a cylinder is 2 feet, and its altitude 6 feet; find its entire surface, including the bases.*

The circumference of its base is

$$2\pi \times 2 = 2 \times 3.1416 \times 2 = 12.5664 \text{ (B. V, P. XVI)};$$

hence the convex surface is

$$12.5664 \times 6 = 75.3984 \text{ (P. I)}.$$

The area of each base is equal to the square of the radius multiplied by 3.1416 (B. V, P. XV), that is, to

$$4 \times 3.1416 = 12.5664,$$

and, therefore, the two bases together are equal to

$$12.5664 \times 2 = 25.1328.$$

The entire surface is equal to the convex surface, increased by the area of the two bases; *i. e.*, to

$$75.3984 + 25.1328 = 100.5312 \text{ square feet.}$$

2.—*The volume of a cylinder, of which the radius of the base is 10 feet, is 6283.2 cubic feet; find the volume of a similar cylinder of which the diameter of the base is 16 feet, and find, also, the altitude of each cylinder.*

Let V denote the volume required; then (P. II, C. 2),

$$V : 6283.2 :: 512 : 1000,$$

hence, $$V = \frac{6283.2 \times 512}{1000} = 3216.9984 \text{ cubic feet.}$$

The area of the base of the first cylinder is

$$3.1416 \times 100 = 314.16 \text{ square feet;}$$

hence the altitude is equal to the volume, 6283.2, divided by the area of the base, 314.16, which is equal to 20 feet. In like manner, the area of the base of the second cylinder is

$$3.1416 \times 64 = 201.0624,$$

and the altitude is equal to

$$\frac{3216.9984}{201.0624} = 16 \text{ feet.}$$

3.—*Two similar cones have the radii of the bases equal, respectively, to 4½ feet and 6 feet, and the convex surface of the first is 667.59 square feet; find the convex surface of the second and the volume of both.*

The circumference of the base of the first is equal to

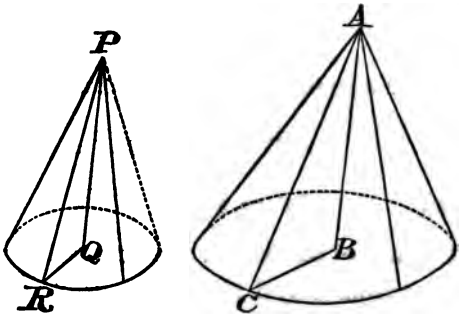
$$2\pi \times 4\tfrac{1}{2} = 2 \times 3.1416 \times 4\tfrac{1}{2} = 26.7036.$$

Since the convex surface is equal to the circumference of the base multiplied by half the slant height (P. III), one-half the slant height is equal to the convex surface divided by the circumference of the base, that is, to

$$\frac{667.59}{26.7036} = 25 \text{ feet.}$$

Since one-half of the slant height is 25 feet, the whole slant height is 50 feet.

Since the cones are similar, the triangles, PQR and ABC, formed by their altitudes, slant heights, and radii of bases, are similar; hence we have



$$QR : BC :: PR : AC;$$

that is, $$4\tfrac{1}{4} : 6 :: 50 : AC;$$

hence, $$AC = \frac{6 \times 50}{4\frac{1}{4}} = 70.588 \text{ feet};$$

which is the slant height of the second cone. The circumference of the base of the second cone is

$$2\pi \times 6 = 37.6992,$$

and hence its convex surface is equal to

$$\frac{37.6992 \times 70.588}{2} = 1330.6 \text{ nearly.}$$

In the second place: In the first cone,

$$PR = 50,$$

and

$$QR = 4.25;$$

hence (B. IV, P. XI. C. 1),

$$\overline{PQ}^2 = \overline{PR}^2 - \overline{QR}^2 = 2500 - 18.0625 = 2481.9375,$$

and $$PQ = \sqrt{2481.9375} = 49.82 \text{ nearly.}$$

In like manner, in the second cone,

$$\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 = 4982.665744 - 36,$$

and $$AB = \sqrt{4946.665744} = 70.33 \text{ nearly.}$$

The area of the base of the first cone is equal to the square of $4\frac{1}{4}$ multiplied by 3.1416 (B. V, P. XV), that is, to

$$18.0625 \times 3.1416 = 56.75 \text{ nearly;}$$

the area of the base of the second cone is, in like manner, equal to

$$36 \times 3.1416 = 113.0976;$$

hence, from (P. V), the volume of the first cone equals

$$56.75 \times \frac{49.82}{3} = 942.4283 \text{ cubic feet,}$$

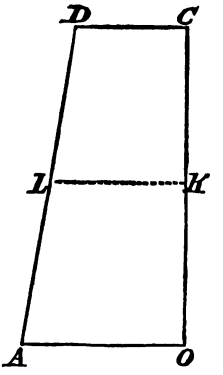
and the volume of the second cone equals

$$113.0976 \times \frac{70.33}{3} = 2651.3847 \text{ cubic feet.}$$

4.—*A line 12 feet long is revolved about another line as an axis; the distance of one extremity of the line from the axis is 4 feet, and of the other extremity 6 feet; find the area of the surface generated.*

Let AD be the revolving line, and OC the axis about which it is revolved. From the middle point L, of the line AD, draw LK perpendicular to OC, and, therefore, parallel to AO and DC. From (P. IV, S.),

$$LK = \frac{AO + DC}{2} = \frac{6 + 4}{2} = 5.$$



The circumference generated by the point L revolving about OC at the distance KL, is equal to

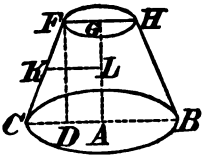
$$2\pi LK = 2 \times 3.1416 \times 5 = 31.416$$

(B. V, P. XVI); hence the area of the surface generated by AD revolving, OC is equal to

$$12 \times 31.416 = 376.992 \text{ square feet (P. IV, S.)};$$

which is the area required.

5.—*Find the convex surface and the volume of the frustum of a cone, the altitude of which is 6 feet, the radius of the lower base being 4 feet and that of the upper base 2 feet.*



Let the figure represent the frustum, the convex surface and the volume of which are required. From F draw FD perpendicular to AC; FD is equal to GA, and AD is equal to FG (B. I, P. XXVIII, C. 2); hence, FD is equal to 6 feet, and AD is equal to 2 feet; since AC is 4 feet, DC must be equal to 2 feet. In the right-angled triangle FDC,

$$\overline{FC}^2 = \overline{FD}^2 + \overline{DC}^2 = 36 + 4 = 40;$$

hence, $$FC = \sqrt{40} = 6.32 \text{ feet nearly.}$$

From K, the middle point of CF, draw KL perpendicular to AG; it will be equal to

$$\frac{AC + GF}{2} = 3 \text{ feet (P. IV, S)};$$

hence the circumference of which LK is the radius is equal to

$$2 \times 3.1416 \times 3 = 18.8496 \text{ (B. V, P. XVI)}.$$

The convex surface of the frustum is equal to CF multiplied by the circumference generated by the point K (P. IV, S); that is, equal to

$$6.32 \times 18.8496 = 119.13 \text{ square feet, nearly.}$$

The volume of the cone which has the circle BC for its base, and an altitude equal to AG, is equal to the area of the base BC, multiplied by $\frac{1}{3}$AG (P. V). The area of the circle BC is equal to $\pi\overline{AC}^2$ (B. V, P. XV), that is, to

$$3.1416 \times 16 = 50.2656 \text{ square feet,}$$

and hence the volume is equal to

$$50.2656 \times 2 = 100.5312.$$

In like manner, the volume of the cone which has the circle HF for its base, and an altitude equal to AG, is equal to the area of the circle HF multiplied by $\frac{1}{3}$AG; that is,

$$\pi \times \overline{GF}^2 \times \tfrac{1}{3}AG = 3.1416 \times 4 \times 2 = 25.1328.$$

A mean proportional between the circles BC and HF is equal to the square root of their product (B. II, P. I), that is, to

$$\sqrt{50.2656 \times 12.5664} = 25.1328;$$

and, therefore, the volume of a cone which has for its base a mean proportional between the circles BC and HF, and an altitude equal to AG, is equal to

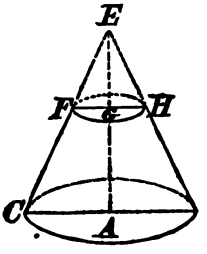
$$25.1328 \times 2 = 50.2656.$$

The volume of the frustum is equal to the sum of the volumes of these three cones (P. VI), that is, to

$$100.5312 + 25.1328 + 50.2656 = 175.9296 \text{ cubic feet};$$

which is the volume required.

6.—*Find the surface and the volume of the cone of which the frustum in the preceding example is a frustum.*



Complete the cone as represented in the figure. The triangles EAC and EGF are similar (B. IV, P. XXI), and, therefore,

$$EA : EG :: AC : GF;$$

hence, $EA \times GF = EG \times AC;$

but $GF = 2,$

and $AC = 4,$ by hypothesis;

therefore, $2EA = 4EG,$

and $EA = 2EG;$

but $EA = EG + GA,$

and, consequently, $2EG = EG + GA,$

and $EG = GA;$

but GA is 6 feet, by hypothesis, and EG must then be 6 feet, and EA, the altitude of the cone, is 12 feet. In the right-angled triangle EAC,

$$\overline{EC}^2 = \overline{EA}^2 + \overline{AC}^2 \text{ (B. IV, P. XI)} = 144 + 16 = 160;$$

hence $$EC = \sqrt{160} = 12.649.$$

The circumference of the base is equal to $2\pi AC$ (B. V, P. XVI); *i. e.*, to

$$2 \times 3.1416 \times 4 = 25.1328;$$

hence the convex surface of the cone equals (P. III)

$$\frac{12.649 \times 25.1328}{2} = 158.95239 \text{ square feet};$$

to this add 50.2656, the area of the base BC, as found in the last example, and the sum will be 209.21799 square feet; which is the area of the surface required.

The volume of the cone is equal to its base, BC, multiplied by one-third its altitude (P. V); *i. e.*, to

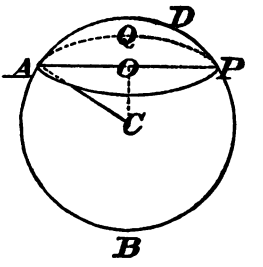
$$50.2656 \times \tfrac{1}{3}^2 = 201.0624 \text{ cubic feet};$$

which is the volume required.

7.—A small circle, the radius of which is 4 feet, is 3 feet from the centre of the sphere; find the circumference of a great circle of the same sphere.

Let ABD be a sphere, AQP the small circle whose radius

OA, is 4 feet, and whose distance from the centre, C, of the sphere, is 3 feet. Draw CO perpendicular to the plane of the circle, it will meet the plane at the centre, O, of the circle (P. VII, C. 3); draw the radius, CA, of the sphere; then in the right-angled triangle AOC,



$$\overline{CA}^2 = \overline{CO}^2 + \overline{OA}^2 = 9 + 16 = 25;$$

hence CA, the radius of the sphere, equals 5, and, therefore, the radius of a great circle of the sphere is 5 feet (P. VII, C. 1), and the circumference is equal to $2\pi \times 5$ (B. V, P. XVI), that is, to

$$2 \times 3.1416 \times 5 = 31.416 \text{ feet};$$

which is the circumference required.

8.—*The radius of a sphere is 10 feet; find the area of a small circle distant from the centre 6 feet.*

The square of the radius of the small circle is equal to the square of 10 diminished by the square of 6 (P. VII, C. 4), that is, to

$$100 - 36 = 64;$$

the area of the circle is (B. V, P. XV) equal to

$$64 \times 3.1416 = 201.0624 \text{ square feet},$$

which is the area required.

9.—*Find the area of the surface generated by the semiperimeter of a regular semihexagon revolving about its axis, the radius of the inscribed circle being 5.2 feet and the axis 12 feet.*

The circumference of the inscribed circle is (B. V, P. XVI) equal to

$$2\pi \times 5.2 = 2 \times 3.1416 \times 5.2 = 32.67264,$$

and the surface generated by the semiperimeter is (P. IX) equal to

$$32.67264 \times 12 = 392.07168 \text{ square feet};$$

which is the area required.

10.—*The area of the surface generated by the semiperimeter of a regular semioctagon revolved about an axis is 52.32 square feet, and the radius of the inscribed circle is 3.62 feet; find the axis.*

The circumference of the inscribed circle is equal to

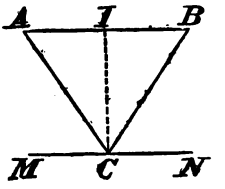
$$2 \times 3.1416 \times 3.62 = 22.745184;$$

since the area of the surface generated is equal to the circumference of the inscribed circle multiplied by the axis (P. IX), the axis must be equal to the area divided by the circumference of the inscribed circle, that is, to

$$\frac{178.2426}{22.745184} = 7.8365 \text{ feet};$$

which is the axis required.

11.—*An isosceles triangle, whose base is 8 feet and altitude 9 feet, is revolved about a line passing through its vertex and parallel to its base; how many cubic feet in the volume generated?*



Let ACB be the triangle and MN the line about which it is revolved; then

vol. CAB $= AB \times 2\pi CI \times \frac{1}{3}CI$ (P. XII)

that is, vol. CAB $= 8 \times 2 \times 3.1416 \times 9 \times 3$

$= 1357.1712$ cubic feet;

which is the volume required.

12.—*The altitude of a zone is 3 feet, and the radius of the sphere is 5 feet; find the area of the zone and the volume of the corresponding spherical sector.*

The circumference of a great circle of the sphere is equal to

$$2\pi \times 5 = 31.416 \text{ (B. V, P. XVI)};$$

hence the area of the zone is equal to

$$31.416 \times 3 = 94.248 \text{ square feet (P. X, C. 2)};$$

and the volume of the corresponding sector is equal to

$$94.248 \times \tfrac{5}{3} = 157.08 \text{ cubic feet (P. XIV, C. 1)}.$$

13.—*Find the surface and the volume of a sphere whose radius is 4 feet.*

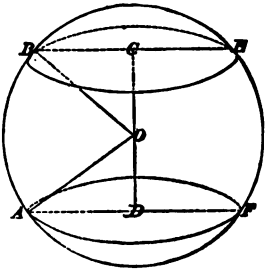
The surface equals (P. X, C. 1),

$$4\pi R^2 = 4 \times 3.1416 \times 16 = 201.0624 \text{ square feet,}$$

and the volume equals (P. XIV),

$$201.0624 \times \tfrac{4}{3} = 268.0832 \text{ cubic feet.}$$

14.—*The radius of a sphere is 5 feet; how many cubic feet in a spherical segment whose altitude is 7 feet and the distance of whose lower base from the centre of the sphere is 3 feet?*



Since the altitude of the segment is greater than the radius of the sphere, the segment will be situated as represented in the figure, and the volume required will be equal to the sum of the volumes of the sector generated by OAB, and the cones generated by the triangles OCB and ODA (P. XIV, S.).

The sector is equal to, (P. XIV, C. 1),

$$2\pi OA \times DC \times \tfrac{1}{3}OA = 2 \times 3.1416 \times 5 \times 7 \times \tfrac{1}{3}$$
$$= 366.52 \text{ cubic feet};$$

the cone generated by OCB is equal to, (P. V),

$$\pi\overline{BC}^2 \times \tfrac{1}{3}OC;$$

but, since DC is 7 feet and OD is 3 feet, OC must be 4 feet, and

$$\overline{BC}^2 = \overline{OB}^2 - \overline{OC}^2 = 25 - 16 = 9;$$

therefore, the volume of this cone is equal to

$$3.1416 \times 9 \times \tfrac{4}{3} = 37.6992 \text{ cubic feet};$$

the cone generated by the triangle ODA is equal to

$$\pi\overline{AD}^2 \times \tfrac{1}{3}OD;$$

but $$\overline{AD}^2 = \overline{OA}^2 - \overline{OD}^2 = 25 - 9 = 16,$$

and the volume of this cone is equal to

$$3.1416 \times 16 \times \tfrac{2}{3} = 50.2656 \text{ cubic feet};$$

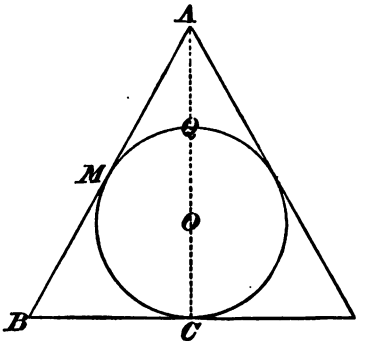
hence the volume of the segment is equal to

$$366.52 + 37.6992 + 50.2656 = 454.4848 \text{ cubic feet};$$

which is the volume required.

15.—*A cone such that the diameter of its base is equal to its slant height is circumscribed about a sphere; show that the surface of the sphere is to the entire surface of the cone, including its base, as 4 is to 9, and that the volumes are in the same ratio.*

Let ACB be a right-angled triangle, in which the base CB is one-half the hypothenuse AB, and let CMQ be a semicircle tangent to AB and CB at M and C; then if the triangle and the semicircle be revolved about AC as an axis, they will generate the cone and the sphere required. Let R denote the radius OC; then



$$AC = 3R,$$

and

$$AB = 2R\sqrt{3} \text{ (B. V, E. 12, Key)},$$

and BC, which equals one-half of AB, is equal to

$$R\sqrt{3}.$$

The convex surface of the cone is equal to the circumference described by BC multiplied by $\frac{1}{2}$AB (P. III), that is, to

$$2\pi BC \times \tfrac{1}{2}AB;$$

to this add the area of the base, or

$$\pi\overline{BC}^2 \text{ (B. V, P. XV)},$$

and we have the whole surface equal to

$$2\pi BC \times \tfrac{1}{2}AB + \pi BC^2 =$$

$$2\pi \times R\sqrt{3} \times \frac{2R\sqrt{3}}{2} + \pi \cdot 3R^2 = 9\pi R^2;$$

that is, the entire surface of the cone is equal 9 great circles of the sphere. The surface of the sphere is equal to 4 great circles (P. X, C. 1); hence the surface of the sphere is to the entire surface of the cone as 4 is to 9; *which was to be proved* in the first place.

The volume of the cone is equal to its base multiplied by one-third its altitude (P. V), that is, to

$$\pi\overline{BC}^2 \times \tfrac{1}{3}AC = 3\pi R^2 \times \frac{3R}{3} = \tfrac{3}{3}\pi R^3;$$

the volume of the sphere is equal to

$$\tfrac{4}{3}\pi R^3 \text{ (P. XIV, C. 2)};$$

hence the volume of the cone is to the volume of the cone as $\frac{3}{3}$ is to $\frac{4}{3}$ or as 4 is to 9; *which was to be proved* in the second place.

16.—*The radius of a sphere is 6 feet; find the entire surface and the volume of the circumscribing cylinder.*

Let R denote the radius of the sphere, then the entire surface of the cylinder is equal to $6\pi R^2$, and the volume of the cylinder is equal to $2\pi R^3$ (P. XV), that is, the surface equals

$$6 \times 3.1416 \times 36 = 678.5856 \text{ square feet,}$$

and the volume is equal to

$$2 \times 3.1416 \times 216 = 1357.1712 \text{ cubic feet.}$$

17.—*A cone, with the diameter of the base and the slant height equal, is circumscribed about a sphere whose radius is 6 feet; find the entire surface and the volume of the cone.*

The entire surface of the cone is equal to 9 great circles of the sphere (E. 15, Key), that is, to

$$9\pi R^2 = 9 \times 3.1416 \times 25 = 706.86 \text{ square feet;}$$

and the volume is equal to $3\pi R^3$ (E. 15, Key), that is, to

$$3 \times 3.1416 \times 125 = 1178.1 \text{ cubic feet.}$$

18.—*A cone, with the diameter of the base and the slant height equal, and a cylinder, are circumscribed about a sphere; what relation exists between the entire surfaces and the volumes of the cylinder, the cone and the sphere?*

From (P. X, C. 1) the surface of the sphere $= 4\pi R^2$.

From (P. XV) the entire surface of the cylinder $= 6\pi R^2$.

From (E. 15, Key) the entire surface of the cone $= 9\pi R^2$.

Hence the surfaces bear to each other the relation expressed by the numbers 4, 6, and 9; and as 6 is a mean proportional between 4 and 9, so is the surface of the cylinder a mean proportional between the surface of the sphere and the surface of the cone.

In the second place,

From (P. XIV, C. 2) the volume of the sphere $= \frac{4}{3}\pi R^3$.

From (P. XV) the volume of the cylinder $= \frac{2}{3}\pi R^3$.

From (E. 15, Key) the volume of the cone $= \frac{2}{3}\pi R^3$.

Hence the volumes bear the same relation to each other as the surfaces, that is the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

19.—*The edge of a regular octaedron is 10 feet, and the radius of the inscribed sphere is 4.08 feet; find the volume of the octaedron.*

Each face of the octaedron being an equilateral triangle with a side of 10 feet, its area is equal to

$$100 \times .433 = 43.3 \text{ square feet (B. IV, E. 15)},$$

and as there are 8 such faces, the entire surface of the octaedron is equal to

$$43.3 \times 8 = 346.4 \text{ square feet};$$

the volume is equal to the surface multiplied by one-third of the radius of the inscribed sphere (P. XV, S.), that is, to

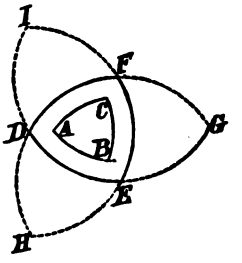
$$346.4 \times \frac{4.08}{3} = 471.104 \text{ cubic feet};$$

which is the volume required.

BOOK IX.

1.—*The sides of a spherical triangle are 80°, 100°, and 110°; find the angles of its supplemental triangle, and the angles of each of its polar triangles.*

Let ABC be the triangle whose sides are given, and let



$$AB = 80°,$$

$$BC = 100°,$$

and $$AC = 110°.$$

Draw the *supplemental* triangle DEF, and prolong its sides, making the three other triangles, DFI, DEH, and EFG, polar to ABC. Each angle of DEF is equal to a semicircumference minus the side lying opposite to it in ABC (P. VI), and as a semicircumference contains 180° (see Note, p. 92*), the angles of DEF are, respectively,

$$180° - 80°,$$

$$180° - 100°,$$

$$180° - 110°;$$

or

$$\text{Angle DFE} = 100°,$$

$$\text{Angle FDE} = 80°,$$

$$\text{Angle DEF} = 70°;$$

which are the angles required.

In the triangle DFI, which is a polar triangle of ABC, but not supplemental (see P. V, S., and P. VI, S.), the angle DIF is equal to the angle DEF, because each is the angle formed by the semicircumferences EFI and EDI, and hence

$$\text{DIF} = 70°;$$

since the arc DF meets the arc EFI, the sum of the spherical angles DFE and DFI is equal to two right angles (P. IV, C. 2), and hence DFI is equal to two right angles diminished by the angle DFE, that is, to

$$180° - 100° = 80°;$$

in like manner, the angle FDI is equal to two right angles diminished by FDE, or to

$$180° - 80° = 100°.$$

In the triangle HDE, polar but not supplemental to ABC, the angle H is equal to the angle DFE, or 100°, as each is contained by the semicircumferences FDH and FEH; angle HED is equal to two right angles diminished by the angle DEF, or to

$$180° - 70° = 110°;$$

angle HDE being vertical to angle FDI, is equal to it (P. IV, C. 2), and hence is equal to 100°.

In the triangle FGE, polar but not supplemental to ABC, the angle G is equal to the angle FDE, being contained by the same semicircumferences DEG and DFG, and hence

$$\text{Angle } G = 80°;$$

angle GEF being vertical to HED, is equal to it (P. IV, C. 2), and hence

$$GEF = 110°;$$

angle EFG, being vertical to DFI, is equal to it, and therefore

$$EFG = 80°.$$

2.—*Find the area of a trirectangular triangle, on a sphere whose diameter is 8 feet.*

Eight trirectangular triangles make up the entire surface of a sphere (P. XIV, C. 3); therefore, one trirectangular is $\frac{1}{8}$ of the surface of the sphere on which it is situated. The surface of a sphere is equal to $4\pi R^2$, R denoting the radius of the sphere (B. VIII, P. X, C. 1), and hence the area of a trirectangular triangle is equal to

$$\frac{4\pi R^2}{8} = \tfrac{1}{2}\pi R^2.$$

In this sphere, $R = 4$;

hence the area of the triangle is equal to

$$\tfrac{1}{2}\pi \times 16 = 3.1416 \times 8 = 25.1328 \text{ square feet};$$

which is the area required.

3.—*Find the area of a trirectangular triangle, on a sphere whose surface and volume may be expressed by the same number.*

Let R denote the radius of the sphere. The surface of the sphere equals $4\pi R^2$ (B. VIII, P. X, C. 1), and the volume equals $\frac{4}{3}\pi R^3$ (B. VIII, P. XIV, C. 2), and since these expressions are numerically equal by hypothesis, we have

$$4\pi R^2 = \tfrac{4}{3}\pi R^3,$$

from which we have $R = 3$.

The area of a trirectangular triangle is equal to $\frac{1}{8}\pi R^2$ (E. 2, Key), *i. e.*, to

$$\tfrac{1}{8} \times 3.1416 \times 9 = 14.1372;$$

which is the area required.

4.—*The angle of a lune, on a sphere whose radius is 5 feet, is 50°; find the area of the lune and the volume of the corresponding wedge.*

The area of a trirectangular triangle on this sphere is, as shown in the preceding example, equal to $\frac{1}{8}\pi R^2$ (R being the radius of the sphere), that is, to

$$\tfrac{1}{8} \times 3.1416 \times 25 = 39.27;$$

the angle of the lune, in terms of the right angle as a unit, is $\frac{5}{9}\frac{0}{0}$, or $\frac{5}{9}$, and twice the angle is $\frac{10}{9}$; therefore, from (P. XV, C. 2), the area of the lune is equal to

$$39.27 \times \tfrac{10}{9} = 43.633 \text{ square feet};$$

which is the area required.

From (P. XV, S.), the volume of the corresponding wedge is equal to

$$43.633 \times \tfrac{5}{3} = 72.72 \text{ cubic feet};$$

which is the volume required.

5.—*The area of a lune is 33.5104 square feet, and the angle of the lune is 60°; find the surface and the volume of the sphere.*

From (P. XV), this lune is to the surface of the sphere as its angle, or 60°, is to 4 right angles, or 360°; that is, as 1 is to 6; hence the surface of the sphere is 6 times the area of the lune, or

$$33.5104 \times 6 = 201.0624 \text{ square feet};$$

which is the surface required.

R being the radius of the sphere, the area of the surface of the sphere is $4\pi R^2$ (P. VIII, C. 1); therefore, in this case,

$$4\pi R^2 = 201.0624,$$

and

$$R^2 = \frac{201.0624}{4 \times 3.1416} = 16,$$

and hence,

$$R = 4.$$

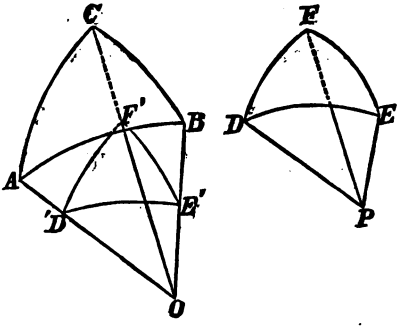
Consequently, the volume of the sphere (B. VIII, P. XIV) is equal to

$$201.0624 \times \tfrac{4}{3} = 268.0832;$$

which is the volume required.

6.—*Show that if two spherical triangles on unequal spheres are mutually equiangular, they are similar.*

Let ABC and DEF be two spherical triangles on the unequal spheres whose centres are O and P, and let them be mutually equiangular; then will they be similar.



Place the centre P on the centre O, and let the

line PD take the direction OA, the point D falling on D', and the plane DPE take the direction of and coincide with the plane AOB, and the point E will then be somewhere in the plane AOB; then since the spherical angles A and D are equal, the diedral angle formed by the planes AOB and AOC, is equal to that formed by the planes DPE and DPF (D. 1); and as the plane DPE is coincident with the plane AOB, the plane DPF will take direction of and, as the point O is common, be coincident with the plane AOC, and the point F will be somewhere in the plane AOC. Since the spherical angles B and E are equal, the diedral angles formed by the planes AOB and BOC is equal to that formed by the planes DPE and EPF; and as the plane DPE takes the direction of and is coincident with the plane AOB, the plane EPF must take the direction of and, as the point O is common, be coincident with the plane BOC, and the points E and F will be somewhere in the plane BOC. As E is at the same time in the planes AOB and BOC, it must be at some point of their intersection, as E'; and as F is at the same time in the planes AOC and BOC, it must be at the same point of their intersection, as F'; and, therefore, the figure P–DFE will take the position O–D'F'E'. As the arcs D'F' and AC subtend the same angle, AOC, at the centre, they are similar and we have

D'F' or DF : AC :: OF' : OC (B. V, P. XIII, C. 2);

for a like reason, F'E' or FE : BC :: OF' : OC;

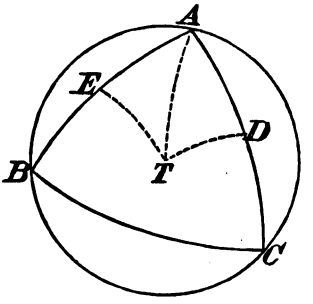
hence (B. II, P. IV), DF : AC :: FE : BC;

in like manner, FE : BC :: DE : AB:

hence, the triangles ABC and DEF are mutually equiangular by

hypothesis, and have their corresponding sides proportional, as just shown, and are, therefore, similar; *which was to be proved.*

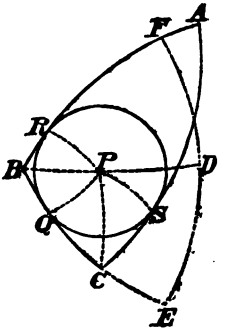
7.—*Show how to circumscribe a circle about a given spherical triangle.*



Let ABC be a spherical triangle. At E, the middle point of AB, draw ET, the arc of a great circle, perpendicular to AB; and at D, the middle point of AC, draw DT, the arc of a great circle, perpendicular to AC; ET and DT will intersect at some point, as T; since T is on a perpendicular to AB at its middle point, it is equally distant from A and B (Gen. S. 1, 2°); and for a like reason T is equally distant from A and C. From T, as a pole, at a distance equal to the arc TA of a great circle, describe a circle (see P. III, S.), it will pass through the three vertices A, B, and C, and will circumscribe the triangle ABC; which was to be done.

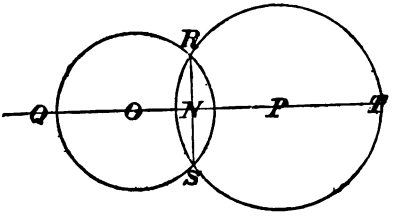
Cor. 1.—If the arcs ET and DT be prolonged, they will intersect in a second point on the surface of the sphere, the two points of intersection being the extremities of a diameter of the sphere (see B. VIII, P. VII, C. 5). The second point of intersection will be the other pole of the circle described from T through the vertices A, B, and C (see P. III, C. 2), and hence the circle described from this second point of intersection as a pole, through the vertices A, B, and C, will coincide with the one described from T as a pole.

8.—*Show how to inscribe a circle in a given spherical triangle.*



Let ABC be the triangle. From B, as a pole, describe the arc of a great circle, EF, included between the sides BA and BC, produced if necessary; EF is the measure of the angle B (P. IV). Draw BD, the arc of a great circle, to the middle point of the arc EF; BD bisects the angle B, as the two parts, EBD and DBF, are each measured by one-half the arc EF. In like manner, bisect the angle C by the arc, CP, of a great circle. From P, the point of intersection of BD and CP, draw the arc, PQ, of a great circle perpendicular to BC; from P, draw the arc, PR, of a great circle, making the angle BPR equal to the angle BPQ; then, the triangles BPR and BPQ have the angle BPR = angle BPQ, and angle PBR = angle PBQ, by construction, and the side BP common, and, hence, the remaining parts are equal (P. IX); *i. e.*, PR = PQ, and angle PRB = angle PQB, and, therefore, PRB is a right angle. From P draw PS, making the angle CPS = angle CPQ; and, as before, it may be shown that PS = PQ, and angle PSC is a right angle; hence, PQ, PR, and PS, are perpendicular respectively to the sides of the triangle and are the shortest distances on the surface of the sphere from P to those sides; therefore if the arc PQ be turned about the point P as a pole, the point Q will describe a circle, RQS, tangent to each of the sides of the triangle and inscribed in it; which was to be done.

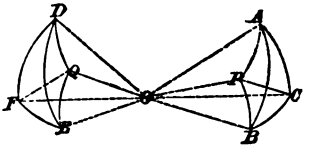
9.—*Show that the intersection of the surfaces of two spheres is a circle, and that the line which joins the centres of two intersecting spheres is perpendicular to the circle in which their surfaces intersect.*



Let RQS and TRS be two circles which intersect at the points R and S; then will their centres, O and P, be in the straight line OP, perpendicular to the line RS, which joins their points of intersection, R and S, at its middle point N (B. III, P. XI). Revolve the two circles about the line QT, through a half revolution; they will generate two spheres which intersect; R and S are the only points common to the circumferences which generate the surfaces of the spheres, and hence the only points common to the two surfaces must lie in the circumference generated by the points R and S moving through the half revolution around QT; *which was to be proved* in the first case.

Since NR = NS in every position through the half revolution, N is the centre of the circumference in which the two surfaces intersect; and since this circumference lies on both spheres, the line OP, joining the centres of the two spheres, passes through the centre of the circle of intersection, and is, therefore, perpendicular to the plane of the circle (B. VIII, P. VII, C. 3); *which was to be proved* in the second case.

10.—*Show that two spherical pyramids of the same or equal spheres, which have symmetrical triangles for bases, are equal in volume.*



Let ABC and DEF be symmetrical triangles situated on a sphere whose centre is O, and let the side DE be equal to the side AB, DF to AC, and EF to BC; then will the spherical pyramids, ABC–O and DEF–O, be equal in volume.

For, conceive a small circle to be drawn through A, B, and C, and let P be its pole; draw arcs of great circles from P to A, B, and C. Draw the arc of a great circle FQ, making the angle DFQ equal to ACP, and lay off on it FQ equal to CP; draw arcs of great circles QD and QE. Now, from (P. XVI), the triangles QFD and PAC, being isosceles and having all their parts equal, may be placed so as to coincide throughout, the side FD falling on AC, DQ on CP, and FQ on AP; hence the two pyramids, QFD–O and PAC–O, may be placed so as to coincide throughout and are therefore equal in volume; in like manner, the triangles QFE and PCB, being isosceles and having all their parts equal, may be placed so as to coincide throughout, and, therefore, the pyramids QFE–O and PCB–O may be placed so as to coincide throughout, and hence are equal in volume; consequently the sum of the pyramids QFD–O and QFE–O, or the pyramid DFEQ–O, is equal to the sum of the pyramids PAC–O and PCB–O, or the pyramid ACBP–O. Again, in like manner as before, from (P. XVI), the triangles QDE and PAB, being isosceles and having all their parts equal, may be placed so as to coincide throughout, and, consequently, the pyramids QDE–O and PAB–O may be placed so as to coincide throughout, and are, therefore, equal in volume. If, then, from the

pyramid DFEQ–O be taken the pyramid QDE–O, and from the pyramid ACBP–O be taken the pyramid PAB–O, the remainders, DEF–O and ABC–O, will be equal in volume.

If the point P falls within the triangle ABC, the point Q will fall within the triangle DEF. In this case the pyramid DEF–O is equal to the sum of the pyramids QFD–O, QFE–O, and QDE–O, and the pyramid ABC–O is equal to the sum of the equal pyramids PAC–O, PBC–O, and PAB–O, and the proposition holds good: therefore, in either case, the spherical pyramids having symmetrical triangles for bases are equal in volume; *which was to be proved.*

11.—*The circumferences of two great circles intersect on the surface of a hemisphere whose diameter is 10 feet, and the acute angle formed by them is 40°; find the sum of the opposite triangles thus formed, and the sum of the corresponding spherical pyramids.*

The sum of the opposite triangles formed is equal to a lune whose angle is 40° (P. XVII). The area of the lune is equal to $\frac{1}{2}\pi R^2$ (the trirectangular triangle) multiplied by $\frac{8}{9}$, twice the angle of the lune in terms of the right angle taken as a unit (see E. 4, Key), that is, to

$$\frac{1}{2} \times 3.1416 \times 25 \times \frac{8}{9} = 34.91 \text{ square feet};$$

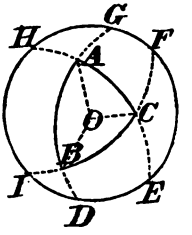
which is the sum of the areas required.

The sum of the corresponding spherical pyramids is equal to the wedge whose angle is 40° (P. XVII, S.). The volume of the wedge (P. XV, S.) is equal to

$$34.91 \times \frac{4}{3} = 58.18 \text{ cubic feet};$$

which is the sum of the pyramids required.

12.—*Show that the volume of a triangular spherical pyramid is equal to its base multiplied by one-third the radius of the sphere.*



Let O–ABC be a triangular spherical pyramid of a sphere whose centre is O; then will its volume be equal to the area of the triangle, ABC, which forms its base, multiplied by one-third of R, R denoting the radius of the sphere.

For, produce the sides of the triangle ABC till they meet the circumference of the great circle DEFG, drawn at pleasure without the triangle; conceive radii drawn from O to the points D, E, F, G, H, and I. From (P. XVII, S.), the sum of the spherical pyramids O–AHG and O–ADE, is equal to the wedge whose angle is A; but the volume of this wedge is equal to

$$2A \times T \times \tfrac{1}{3}R \text{ (P. XV, C. 2, and S.).}$$

In like manner, it may be shown that the sum of the pyramids O–BFG and O–BID is equal to

$$2B \times T \times \tfrac{1}{3}R,$$

and that the sum of the pyramids O–CIH and O–CFE is equal to

$$2C \times T \times \tfrac{1}{3}R.$$

But the sum of the volumes of these six spherical pyramids exceeds the volume of the hemisphere by twice the pyramid O–ABC. Now the volume of the hemisphere, being half the volume of the sphere, is its equal to its surface, which is half the surface of the sphere, multiplied by one-third the radius, or to

$$4\,T \times \tfrac{1}{3}R;$$

hence we shall have

$$2 \times \text{Vol. O–ABC} = 2A \times T \times \tfrac{1}{3}R + 2B \times T \times \tfrac{1}{3}R + 2C \times T \times \tfrac{1}{3}R - 4T \times \tfrac{1}{3}R;$$

or, by reducing and factoring,

$$\text{Vol. O–ABC} = (A + B + C - 2) \times T \times \tfrac{1}{3}R;$$

but

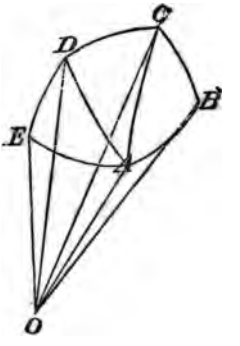
$$(A + B + C - 2) \times T$$

equals the area of the triangle ABC (P. XVIII); hence,

$$\text{Vol. O–ABC} = \text{Area ABC} \times \tfrac{1}{3}R;$$

which was to be proved.

13.—*Show that the volume of any spherical pyramid is equal to its base multiplied by one-third the radius of the sphere.*



Let ABCDE–O be a spherical pyramid of the sphere whose centre is O; through the vertex A and each of the diagonally opposite vertices, D and C, of the polygon ABCDE, forming the base of the pyramid, pass planes which shall also pass through the centre O; these planes will divide the given pyramid into triangular pyramids; the volume of each triangular pyramid is equal to its base multiplied by one-third the radius of the sphere, as just shown; hence the sum of the volumes of the triangular pyramids, or the volume of the given pyramid, is equal to the sum of their bases, or the base of

the given pyramid, multiplied by one-third the radius of the sphere; *which was to be proved.*

14.—*Find the volume of a spherical pyramid whose base is a trirectangular triangle, the diameter of the sphere being 8 feet.*

The trirectangular triangle being one-eighth the surface of the sphere (P. XIV, C. 3), is equal to $\frac{1}{8}$ of $4\pi R^2$ (B. VIII, P. X, C. 1), that is, to

$$\frac{1}{8} \times 4 \times 3.1416 \times 16 = 25.1328 \text{ square feet};$$

and, from (E. 12, Key), the volume of the pyramid is equal to

$$25.1328 \times \frac{4}{3} = 33.5104 \text{ cubic feet};$$

which is the volume required.

15.—*The angles of a triangle, on a sphere whose radius is 9 feet, are 100°, 115°, and 120°; find the area of the triangle and the volume of the corresponding spherical pyramid.*

The sum of the angles of the triangle is 335°; this sum diminished by two right angles, or 180°, is 155°, which is the spherical excess (P. XIV, S.); $\frac{155}{90}$ is this spherical excess expressed in terms of the right angle as a unit. A trirectangular triangle of the sphere is equal to $\frac{1}{2}\pi R^2$ (see E. 2, Key), that is, to

$$\frac{1}{2} \times 3.1416 \times 81 = 127.2348;$$

hence the area of the triangle is (P. XVIII) equal to

$$127.2348 \times \frac{155}{90} = 219.1266 \text{ square feet};$$

which is the area required.

The volume of the corresponding spherical pyramid is (E. 12, Key) equal to

$$219.1266 \times \tfrac{3}{8} = 657.3798 \text{ cubic feet};$$

which is the volume required.

16.—*A spherical pyramid, of a sphere whose diameter is 10 feet, has for its base a triangle of which the angles are 60°, 80°, and 85°; what is its ratio to a pyramid whose base is a trirectangular triangle of the same sphere?*

The two pyramids are to each other as their bases (P. XVIII, S. 1); the spherical excess of the base of the first pyramid is (P. XIV, S.) equal to

$$60° + 80° + 85° - 180° = 45°;$$

and this, expressed in terms of the right angle as a unit, is equal to $\frac{4}{8}\frac{5}{0}$, or $\frac{1}{2}$; hence the area of the base of the first pyramid is, if T denote a trirectangular triangle of the sphere, equal to $\frac{1}{2} \times T$ (P. XVIII), the area of the base of the other pyramid is, by hypothesis, equal to T; therefore, the first pyramid is to the second as $\frac{1}{2}$T is to T, or as $\frac{1}{2}$ is to 1; which is the ratio required.

17.—*The sum of the angles of a regular spherical octagon is 1140°, and the radius of the sphere is 12 feet; find the area of the octagon.*

The spherical excess of the octagon is (P. XIV, S.) equal to 1140° diminished by two right angles taken 6 times, or 12 right angles, that is, to

$$1140° - 1080° = 60°;$$

this expressed in terms of the right angle as a unit, is $\frac{6}{9}\frac{0}{0}$ or $\frac{2}{3}$;

a trirectangular triangle of this sphere is equal to $\frac{1}{8}\pi R^2$, or to

$$\frac{1}{8} \times 3.1416 \times 144 = 228.1952;$$

and the area of the octagon is (P. XIX) equal to

$$228.1952 \times \tfrac{2}{3} = 152.1301 \text{ square feet};$$

which is the area required.

18.—*The volume of a spherical pyramid, whose base is an equiangular triangle, is 84.8232 cubic feet, and the radius of the sphere is 6 feet; find one of the angles of the base.*

The volume of the pyramid is equal to its base multiplied by one-third the radius of the sphere (E. 12, Key); the base is equal to its spherical excess, expressed in terms of the right angle taken as a unit, multiplied by the trirectangular triangle (P. XVIII); the trirectangular triangle, the radius of the sphere being R, is equal to $\frac{1}{8}\pi R^2$ (E. 2, Key); hence the volume of the pyramid is (letting E represent the spherical excess of the base) equal to

$$E \times \tfrac{1}{8}\pi R^2 \times \tfrac{1}{3}R = E \times \tfrac{1}{6}\pi R^3;$$

therefore, the spherical excess, or

$$E = \frac{\text{Volume of pyramid}}{\frac{1}{6}\pi R^3} = \frac{84.8232}{\frac{1}{6} \times 3.1416 \times 216} = \frac{3}{4};$$

that is, the spherical excess expressed in terms of the right angle as a unit, is $\frac{3}{4}$, and hence that excess, expressed in degrees, is $\frac{3}{4}$ of $90° = 67\frac{1}{2}°$; therefore, since the spherical excess of a triangle is equal to the sum of its angles diminished by two right angles, or 180° (P. XIV, S.), the sum of the angles is equal to the spherical excess increased by 180°, that is, to

$$67\tfrac{1}{2}^\circ + 180^\circ = 247\tfrac{1}{2}^\circ;$$

and as the base is an equiangular triangle, each angle is equal to

$$\frac{247\frac{1}{2}^\circ}{3} = \frac{495^\circ}{6} = 82\tfrac{1}{2}^\circ;$$

which is the angle required.

19.—*Given a spherical angle of 40°; what is the number of degrees in the longest arc of a great circle that can be drawn perpendicular to either side of the angle and included between the two sides?*

Since the given angle is acute, its measure, or 40°, is the longest arc (Gen. S. 2, 1°).

20.—*Given a spherical angle of 115°; what is the number of degrees in the shortest arc of a great circle that can be drawn perpendicular to either side of the angle and included between the two sides?*

Since the given angle is obtuse, its measure, or 115°, is the shortest arc (Gen. S. 2, 2°).